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# ON TACHIBANA CONHARMONIC RECURRENT SPACES OF SECOND ORDER

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## ABSTRACT

Lichnerowicz (1960) called a Riemannian space satisfying $R_{ijk,lm} - \lambda_{lm} R^h_{ijk} = 0$, where $\lambda_{lm}$ is a non-zero tensor, a recurrent space of second order. Further, Singh (1971-72) defined Kaehlerian recurrent, Kaehlerian Ricci-recurrent, Kaehlerian projective recurrent spaces of second order and that with Bochner curvature and obtained relations existing between them. In the present paper, we have defined and studied Tachibana conharmonic recurrent spaces of second order and several theorems have been investigated.

**Keywords:** *Tachibana, Conharmonic, Recurrent Spaces*

## INTRODUCTION:

An n (=2m) dimensional Kaehlerian space $K_n$ is a Riemannian space, which admits a structure tensor field $F^h_i$ satisfying the relations:

$$F^h_j F^i_h = -\delta^i_j \,^{*)}, \qquad \text{....(1.1)}$$

$$F_{ij} = -F_{ji}, \ (F_{ij} = F^a_i g_{aj}) \qquad \text{....(1.2)}$$

and

$$F^h_{i,j} = 0, \qquad \text{....(1.3)}$$

where the comma ( , ) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor $g_{ij}$ of the Riemannian space.

The Riemannian curvature tensor, which we denote by $R^h_{ijk}$, is given by

$$R^h_{ijk} = \partial_i \{^{\ h}_{j\ k}\} - \partial_j \{^{\ h}_{i\ k}\} + \{^{\ l}_{j\ k}\}\{^{\ h}_{i\ l}\} - \{^{\ h}_{j\ l}\}\{^{\ l}_{i\ k}\} \,^{**)},$$

*) *All latin indices run over the same range from 1 to n.*

**) $\partial_i \equiv \partial/\partial x^i$, *where* $\{x^i\}$ *denotes real local co-ordinates.*

whereas the Ricci-tensor and the scalar curvature are respectively given by

$$R_{ij} = R^h_{hij} \text{ and } R = R_{ij} g^{ij}.$$

It is well known that these tensors satisfy the identities (Tachibana, 1967).

$$F^a_i F^j_a = R^a_i F^j_a \qquad \text{....(1.4)}$$

$F^a_i R_{aj} = - R_{ia} F^a_j$. .... (1.4)

In view of (1.1), the relation (1.4) gives

$F^a_i R^b_a F^i_b = - R^j_i$. .... (1.5)

Also, multiplying (1.4)ì by $g^{ij}$, we obtain

$F^a_i R^i_a = - R^j_a F^a_j$,

which implies

$F^a_i R^a_i = 0$. ..... (1.6)

If we define a tensor $S_{ij}$ by

$S_{ij} = F^a_i R_{aj}$, .... (1.7)

we have

$S_{ij} = -S_{ji}$. .... (1.8)

Now, an almost Tachibana space is an almost Hermite space, (F, g), where F is an almost complex structure and g is the Hermite metric, such that

$F^h_{i,j} + F^h_{j,i} = 0$. ..... (1.9)

In an almost Tachibana space, we have (Tachibana 1967)

$N^h_{j,i} = -4 (F^a_{i,j}) F^h_a$, ... (1.10)

where $F^h_{i,j}$ is pure in j and i, and N is the Nijenhuis tensor (Yano, 1965).

When the Nijenhuis tensor vanishes, the almost Tachibana space is called a Tachibana space or, in brief, an $T_n$ – space.

A Tachibana space $T_n$ is said to be Tachibana recurrent space (Lal and Singh, 1971), if its curvature tensor field satisfies the condition:

$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0$, .... (1.11)

where $\lambda_a$ is a non-zero vector and is known as recurrence vector field.

The H-conharmonic curvature tensor $T^h_{ijk}$, projective curvature tensor $P^h_{ijk}$, and the Bochner curvature tensor $B^h_{ijk}$ are respectively given by (Sinha 1973)

$$T^h_{ijk} = R_{ijk} + \frac{1}{n+4} [R_{ik} \delta^h_j - R_{jk} \delta^h_i + g_{ik} R^h_j - g_{jk} R^h_i + S_{ik} F^h_j - S_{jk} F^h_i$$

$$+ F_{ik} S^h_j - F_{jk} S^h_i + 2S_{ij} F^h_k + 2F_{ij} S^h_k], \quad \text{... (1.12)}$$

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2} (R_{ik} \delta^h_j - R_{jk} \delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2S_{ij} F^h_k) \quad \text{..... (1.13)}$$

that it is Tachibana 2-Ricci-recurrent.

**Proof**: Differentiating (1.12) covariantlywith respect to $x^l$ and $x^m$ successively, we get

$$T^h_{ijk'lm} = R^h_{ijk'lm} + \frac{1}{n+4}(\delta^h_j R_{ik'lm} - \delta^h_i R_{jk'lm} + g_{ik} R^h_{j'lm} - g_{jk} R^h_{i'lm} + F^h_j S_{ik'lm}$$

$$- F^h_i S_{jk'lm} + F_{ik} S^h_{j'lm} - F_{jk} S^h_{i'lm} + 2F^h_k S^h_{i'lm} + 2F_{ij} S^h_{k'lm}). \quad \ldots (2.9)$$

Multiplying (1.12) by $\lambda_{lm}$ and subtracting the resulting equation from (2.9), we get

$$T^h_{ijk'lm} - \lambda_{lm} T^h_{ijk} = R^h_{ijk'lm} - \lambda_{lm} R^h_{ijk} + \frac{1}{(n+4)}[\delta^h_j (R_{ik'lm} - \lambda_{lm} R_{ik})$$
$$+ \delta^h_i (R_{jk'lm} - \lambda_{lm} R_{jk}) + g_{ik} (R^h_{j'lm} - \lambda_{lm} - \lambda_{lm} R^h_j) - g_{jk} (R^h_{i'lm} - \lambda_{lm} R^h_i)$$
$$+ F^h_j (S_{ik'lm} - \lambda_{lm} S_{ik}) - F^h_i (S_{jk'lm} - \lambda_{lm} S_{jk}) + F_{ik} (S^h_{j'lm} - \lambda_{lm} S^h_j)$$
$$-F_{jk} (S^h_{i'lm} - \lambda_{lm} S^h_i) + 2F^h_k (S_{ij'lm} - \lambda_{lm} S_{ij}) + 2F_{ij} (S^h_{k'lm} - \lambda_{lm} S^h_k)]. \quad .. (2.10)$$

Let the space be Tachibana 2-Rcci-recurrent, then (2.10), in view of (1.7) and (1.16), yields

$$T^h_{ijk'lm} - \lambda_{lm} T^h_{ijk} = R^h_{ijk'lm} - \lambda_{lm} R^h_{ijk}, \quad .. (2.11)$$

which shows that the Tachibana conharmonic 2-recurrent space is Tachibana 2-recurrent.

By virtue of equations (1.2), (1.16), (2.1) and (2.10), we can easily prove the following:

**Corollary (2.1)**. If in a $T_n$, any two of the following properties are satisfied, then the third is also satisfied:

(a) the space is Tachibana 2-recurrent,

(b) the space is Tachibana 2-Ricci-recurrent,

(c) the space is Tachibana conharmonic 2-recurrent.

**Theorem (2.8)** : The necessary and sufficient conditions that a $T_n$ be Tachibana 2-Ricci recurrent, is that

$$T^h_{ijk'lm} - \lambda_{lm} T^h_{ijk} = R^h_{ijk'lm} - \lambda_{lm} R^h_{ijk} \quad \ldots (2.12)$$

**Proof** : Let $T_n$ be Tachibana 2-Ricci-recurrent then, the relation (1.16) is satisfied and so the equation (2.10) reduces to (2.12).

Conversely, if in a $T_n$, (2.12) is satisfied then the equation (2.10) yields

$$\delta^h_j (R_{ik'lm} - \lambda_{lm} R_{ik}) - \delta^h_i (R_{jk,lm} - \lambda_{lm} R_{jk}) + g_{ik} (R^h_{j'lm} - \lambda_{lm} R^h_j) - g_{jk} (R^h_{i'lm}$$
$$-\lambda_{lm} R^h_i) + F^h_i (S_{ik'lm} - \lambda_{lm} S_{ik}) = F^h_j (S_{jk'lm} = \lambda_{lm} S_{jk}) + F_{ik} (S^h_{j'lm} - \lambda_{lm} S^h_j)$$
$$- F_{jk} (S^h_{i'lm} - \lambda_{lm} S^h_i) + 2F^h_k (S_{ij'lm} - \lambda_{lm} S_{ij}) + 2F_{ij} (S^h_{k'lm} - \lambda_{lm} S^h_k) = 0 \quad \ldots (2.13)$$

Simplifying with the help of equations (1.4), (1.5), (1.6) and (1.7) and also using the identity $F_{ij}g^{ij} = 0$, we get

$$(n+2)(R_{ik'lm} - \lambda_{lm} R_{ik}) = 0,$$

which gives $R_{ik'lm} - \lambda_{lm} R_{ik} = 0$,

because $n \neq -2$.

This shows that the space $T_n$ is Tachibana 2-Ricci-recurrent. This completes the proof.

The following theorem is immediate from (2.11):

**Theorem (2.9)**: The necessary and sufficient condition for a Tachibana 2-Ricci-recurrent space to be Tachibana 2-recurrent is that the space be Tachibana conharmonic 2-recurrent.

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