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F-version Of Ciric Quasi-Contractions And Generalization Of Sehgal's Result

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Abstract: In this paper, we answer two open problems regarding fixed points for the $\mathcal{F}$-version of Ciric quasi-contractions and the generalization of Sehgal's result raised by Fabino et al. associated with the mapping $\mathcal{F}$.

Keywords: Wardowski F-contraction; Sehgal type result; Ciric quasi-contraction; fixed point.

1. Introduction

Stefan Banach 1922, formulated the Banach contraction principle (BCP), a landmark concept in the realm of fixed point theory. This principle asserts that any Banach contraction self-map $\mathfrak{T}$ acting on a complete metric space(MS) $(U, \mathfrak{d})$ possesses a unique fixed point(FP). Because of its wide applicability, it has various generalizations, extensions, and applications given by eminent mathematicians (see, for instance, Bryant 1968, Ćirić 1974, Ćirić 1971, Gangwar et.al. 2024, Sehgal 1969). Ciric 1974, provided a significant generalization that has been investigated by many authors (see: Olerio et. Al. 2014 , Parvaneh et.al. 2017). A generalization of BCP on iterated mappings is given by W. Bryant 1968, which is further generalized by Sehgal 1969.

Wardowski 2012, proved a generalization of BCP in the frame of dynamic processes. Let $\mathcal{F}: (0, \infty) \to \mathbb{R}$ comply with the following stipulations:

$(P1)$ The function $\mathcal{F}$ is consistently increasing;

$(P2)$ A sequence $\{t_n\} \in (0, \infty)$ tends to zero if and only if $\mathcal{F}(t_n) \to -\infty$ as $n \to \infty$;

(P3) $\lim_{t \to 0^+} t^k \mathcal{F}(t) = 0$ for some $k \in (0,1)$.

Some examples of such $\mathcal{F}$ are the followings.

Example 1.1. Define the mappings from $(0, \infty)$ into $\mathbb{R}$: $\mathcal{F}_1(t) = e^t, \mathcal{F}_2(t) = -\frac{1}{t}, \mathcal{F}_3(t) = \log t, \mathcal{F}_4(t) = -\frac{1}{\sqrt{t}}, \mathcal{F}_5(t) = \log t$. Then

(i) $\mathcal{F}_1$ fulfills condition $(P1)$ and $(P3)$;

(ii) $\mathcal{F}_2$ and F_4 fulfills condition $(P1)$ and $(P2)$;

(iii) $\mathcal{F}_3$ and $\mathcal{F}_5$ fulfills all three properties.

The set of mappings $\mathcal{F}: (0, \infty) \to \mathbb{R}$ that fulfill the conditions $(P1), (P2)$ and $(P3)$ is called Wardowski's collection $(\mathbb{F})$. Using these functions, we have a new type of contraction on a MS.

Definition 1.2. Let $\mathcal{F} \in \mathbb{F}$, i.e., $\mathcal{F}$ fulfills $(P1), (P2), (P3)$ and let $\mathfrak{T}$ be a self-map on a MS $(\Omega, \mathfrak{d})$. If there exists a positive number $\xi > 0$ for all $\pi, b \in \Omega$ for which $\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}b) > 0$ such that

$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}b)) \leq \mathcal{F}(\mathfrak{d}(\pi, b))\#(1)$

holds. Then the mapping $\mathfrak{T}$ is called an $\mathcal{F}$-contraction introduced by Wardowski 2012.

Let $\mathfrak{T}$ be a $\mathcal{F}$-contraction self-map on a complete MS $(\Omega, \mathfrak{d})$. Then $\mathfrak{T}$ has a unique FP [Fabiano et.al. 2022, Theorem 1.4]. This was the main result of Wardowski in 2012. F-contractions can be important in various fields, including mathematical analysis and applied mathematics, especially in proving the existence of solutions to certain types of equations.

In the past few years, Wardowski's -contraction has many generalizations and extended results given by eminent mathematicians (see, for instance, Fabiano et al., 2020; Mitrović et al., 2021; Vujaković et al. 2020).

Inspired by these results, we answer two open questions introduced by Fabiano et. al. associated with mappings $\mathcal{F}$s that fulfil condition $(P1)$ and $(P2)$ only and provide some examples.

2. Preliminaries

In this section, we give some preliminary and already known results. We use the following result to give positive answer to [Fabiano et al. 2022, Question 4.2].

Theorem 2.1. [Fabiano et al. 2022, Theorem 2.3] Let $\mathfrak{T}$ be a self-map on a complete MS $(\Omega, \mathfrak{d})$. If there exists a function $\mathcal{F}: (0, \infty) \to \mathbb{R}$ that fulfils (P1) and a positive number $\xi > 0$ for all $\pi, b \in \Omega$ for which $\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}b) > 0$ such that

$$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}b)) \leq \mathcal{F}(\mathfrak{d}(\pi, b))\#(2)$$

holds. Then, there exists unique FP of $\mathfrak{T}$.

In [3], Ciric has given the following definition of quasi-contraction, we call it Ciric quasi-contraction.

Definition 2.2. [cirić 1974, A self-map $\mathfrak{T}: \Omega \to \Omega$ on a MS is a Ciric quasi-contraction if there is a real number $q(0 \leq q < 1)$, such that

$\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}b) \leq q \cdot \max\{\mathfrak{d}(\pi, b), d(\pi, \mathfrak{T}\pi), \mathfrak{d}(\pi, \mathfrak{T}b), \mathfrak{d}(\pi, \mathfrak{T}b), \mathfrak{d}(b, \mathfrak{T}\pi)\}$

holds, for every $\pi, b \in \Omega$

We shall use the following two Lemmas in the sequel.

Lemma 2.3. [Fabiano et.al. 2020, Lemma 5] Let $\mathfrak{T}$ be a self-map on a MS $(\Omega, \mathfrak{d})$, and $\{\pi_n\}$ be a picard sequence $\pi_n = \mathfrak{T}\pi_{n-1}, n \in \mathbb{N}$. If $\mathfrak{d}(\pi_{n+1}, \pi_n) < \mathfrak{d}(\pi_n, \pi_{n-1})$ holds for each $n \in N$, then $\pi_n \neq \pi_m$ whenever $n \neq m$.

Lemma 2.4. [Radenović et. Al. 2017, Lemma 4] Let $\{\pi_n\}$ be a non-Cauchy sequence in a MS $(\Omega; \mathfrak{d})$ such that $\lim_{n\to\infty} \mathfrak{d}(\pi_n; \pi_{n+1}) = 0$. Then there exist a real number $\theta > 0$ and two sequences $\{n_{1_k}\}$ and $\{n_{2_k}\}$ with $n_{1_k} > n_{2_k} > k > 0$, such that for $k \to \infty$ the following tend to θ

$\mathfrak{d}\left(\pi_{n_{2_k}}, \pi_{n_{1_k}}\right); \mathfrak{d}\left(\pi_{n_{2_k}}, \pi_{n_{1_k}+1}\right); \mathfrak{d}\left(\pi_{n_{2_k}-1}, \pi_{n_{1_k}}\right); \mathfrak{d}\left(\pi_{n_{2_k}-1}, \pi_{n_{1_k}+1}\right); \mathfrak{d}\left(\pi_{n_{2_k}+1}, \pi_{n_{1_k}+1}\right); \ldots$

3.Main Results:

We need the following definition in order to state the open question.

Definition 3.1. Let $(\Omega, \mathfrak{d})$ be a complete MS. A self-map $\mathfrak{T}$ on $(A, \mathfrak{d})$ is $\mathcal{F}$-version of Ciric's quasi type contraction if there exists $\mathcal{F}$ that fulfils the properties $(P1)$ and $(P2)$ such that

$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}\pi)) \leq \mathcal{F}(\max\{\mathfrak{d}(\pi, b), \mathfrak{d}(\pi, \mathfrak{T}a), \mathfrak{d}(b, \mathfrak{T}b), \mathfrak{d}(\pi, \mathfrak{T}b), \mathfrak{d}(b, \mathfrak{T}\pi)\})\#(3)$

for all $\pi, b \in \Omega$ and for some $\xi > 0$ with $\mathfrak{T}\pi \neq \mathfrak{T}b$.

Note that our $\mathcal{F}$ fulfills $(P1)$ only and $(P2)$.

Now, we state the question introduced by Fabiano et al. associated with the mapping $\mathcal{F}$ which satisfies conditions $(P1)$ only and $(P2)$.

Question 3.2. [Fabiano et.al. 2022, Question 4.1] Let $\mathfrak{T}$ be a $\mathcal{F}$-Ciric's quasi type contraction on a complete MS $(\Omega, \mathfrak{d})$. Does there exist unique FP of $\mathfrak{T}$?

The following example shows that the Question 3.2 has negative answer.

Example 3.3. Let $\Omega = [0,1]$ with usual metric d. Then $(\Omega, \mathfrak{d})$ is a complete MS. Define mappings $\mathfrak{T}: \Omega \to \Omega$ as

$$\mathfrak{T}\pi = \begin{cases} 1 & \text{if } \pi \in (0,1) \\ \frac{1}{2} & \text{if } \pi = 1 \end{cases}$$

Case(i): Let $\pi, b \in [0,1)$ or $\pi = b = 1$. Then we have,

$\mathfrak{T}\pi = \mathfrak{T}b.$

Case(ii): Let $\pi \in [0,1), b = 1$. Then $\mathfrak{T}\pi = 1$ and $\mathfrak{T}b = \frac{1}{2}$. Then $\mathfrak{d}(\mathfrak{T}\pi, Tb) = \left|1 - \frac{1}{2}\right| = \frac{1}{2}$, also $\max\{\mathfrak{d}(\pi,1), \mathfrak{d}(\pi, \mathfrak{T}a), \mathfrak{d}(1, \mathfrak{T}1), \mathfrak{d}(\pi, \mathfrak{T}1), \mathfrak{d}(1, \mathfrak{T}\pi)\} = 1$

. Let $\mathcal{F}: (0, \infty) \to \mathbb{R}$ defined by

$$\mathcal{F}(t) = \begin{cases} -2 & \text{if } t \in (0,1) \\ -\frac{1}{t} & \text{if } t \in [1, \infty) \end{cases}$$

and $\xi = \log 2$.

Then we get

$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi, \mathfrak{T}1)) \le \mathcal{F}(\max\{\mathfrak{d}(\pi,1), \mathfrak{d}(\pi, \mathfrak{T}\pi), \mathfrak{d}(1, \mathfrak{T}1), \mathfrak{d}(\pi, \mathfrak{T}1), \mathfrak{d}(1, \mathfrak{T}\pi)\})$

Hence, $\mathfrak{T}$ fulfils all the conditions of given Question 3.2, despite this $\mathfrak{T}$ has no FPs.

Example 3.3 shows that $\mathcal{F}$-version of Ciric's quasi type contraction map need not have fPs but under certain assumptions, we have the following result. Moreover, in general in a MS $(\Omega; \mathfrak{d})$, it is not true that if $\lim_{n\to\infty} \mathfrak{d}(\pi_n, \pi_{n+1}) = 0$ then $\{\pi_n\}$ is Cauchy but the Picard sequence $\{\pi_n\} = \mathfrak{T}\pi_{n-1} = \mathfrak{T}^n\pi_0$, where $\mathfrak{T}$ is $\mathcal{F}$-version of Ciric's quasi type contraction as mentioned in the following Theorem, is Cauchy if $\lim_{n\to\infty} \mathfrak{d}(\pi_n, \pi_{n+1}) = 0$.

Theorem 3.4. Let $\mathfrak{T}$ be a $\mathcal{F}$-version of Ciric's quasi type contraction on a complete MS $(\Omega, \mathfrak{d})$. Suppose $\mathcal{F}$ and $\mathfrak{T}$ are continuous and for $a_0 \in U$ the picard sequence $\pi_n = \mathfrak{T}\pi_{n-1} = \mathfrak{T}^n\pi_0$ fulfils $\mathfrak{d}(\pi_{n-1}, \pi_{n+1}) < \mathfrak{d}(\pi_n, \pi_{n-1})$ for all $n \in \mathbb{N}$. Then

(i) the Picard sequence $\{\pi_n\}$ converges to some $\pi^* \in \Omega$.

(ii) π^* is a unique FP of $\mathfrak{T}$.

Proof. Since there is no $n_0 \in \mathbb{N}$ such that $\pi_{n_0} = \pi_{n_0 - 1}$ therefore $\pi_n \neq \pi_{n-1}$ forall $n \in \mathbb{N}$. Then $\mathfrak{d}(\pi_n, \pi_{n-1}) > 0, \forall\, n \in \mathbb{N}$. Then from (3), we have

$$\begin{aligned} \xi + \mathcal{F}(\mathfrak{d}(\pi_n, \pi_{n+1})) &= \xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi_{n-1}, \mathfrak{T}\pi_n)) \\ &\le \mathcal{F}(\max\{\mathfrak{d}(\pi_n, \pi_{n-1}), \mathfrak{d}(\pi_n, \mathfrak{T}\pi_n), \mathfrak{d}(\pi_{n-1}, \mathfrak{T}\pi_{n-1}), \mathfrak{d}(\pi_n, \mathfrak{T}\pi_{n-1}), \mathfrak{d}(\pi_{n-1}, \mathfrak{T}\pi_n)\}) \\ &= \mathcal{F}(\max\{\mathfrak{d}(\pi_n, \pi_{n-1}), d(\pi_n, \pi_{n+1}), \mathfrak{d}(\pi_{n-1}, \pi_{n+1})\}) \\ &\le \mathcal{F}(\max\{\mathfrak{d}(\pi_n, \pi_{n-1}), \mathfrak{d}(\pi_n, \pi_{n+1})\}). \end{aligned}$$

If there exists $n \in \mathbb{N}$ such that $\max\{\mathfrak{d}(\pi_n, \pi_{n-1}), d(\pi_n, \pi_{n+1})\} = \mathfrak{d}(\pi_n, \pi_{n+1})$ then we have,

$\xi + \mathcal{F}(\mathfrak{d}(\pi_n, \pi_{n+1})) \le \mathcal{F}(\mathfrak{d}(\pi_n, \pi_{n+1})),$

which is a contradiction. So we have $\max\{\mathfrak{d}(\pi_n, \pi_{n-1}), d(\pi_n, \pi_{n+1})\} = \mathfrak{d}(\pi_n, \pi_{n-1})$. Then

$\mathcal{F}(\mathfrak{d}(\pi_n, \pi_{n+1})) \le \mathcal{F}(\mathfrak{d}(\pi_n, \pi_{n-1})) - \xi. \#(4)$

This implies
$\mathcal{F}(\mathfrak{d}(\pi_n,\pi_{n+1})) \le \mathcal{F}(\mathfrak{d}(\pi_1,\pi_0)) - n\xi.$
Taking $n\to\infty$, we get $\mathcal{F}(\mathfrak{d}(\pi_n,\pi_{n+1})) \to -\infty$. By property $(P2)$, $\lim_{n\to\infty}\mathfrak{d}(\pi_n,\pi_{n+1}) = 0$. Suppose that $\{\pi_n\}$ is not a Cauchy sequence . Then by Lemma 2.4 there are sequences $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k$, for all $k\in\mathbb{N}$ such that the sequences in Lemma 2.4 goes to θ as $k\to\infty$. Also by Lemma 2.3, $\pi_{n_k}\neq\pi_{m_k}$.
Now, from the continuity of $\mathcal{F}$ and inequality 3. we have,
$\mathcal{F}(\theta) \le \mathcal{F}(\theta) - \xi$
This gives a contradiction as $\xi > 0$. Hence, the sequence $\{\pi_n\}$ is Cauchy. By completeness of the MS $(\Omega,\mathfrak{d})$ there exists an element $a^*\in U$ such that
$\lim_{n\to\infty}\pi_n = \pi^*$
Using continuity of $\mathfrak{T}$, we get $\pi^* = \mathfrak{T}\pi^*$. Hence π^* is FP of $\mathfrak{T}$.
Uniqueness: Suppose $\mathfrak{T}$ has two fixed points $\pi^* = \mathfrak{T}\pi^*$ and $b^* = \mathfrak{T}b^*$. Then from (3), we have
$\xi + \mathcal{F}(\mathfrak{d}(\pi^*,b^*)) \le \mathcal{F}(\mathfrak{d}(\pi^*,b^*))$
which gives a contradiction for $\xi > 0$. Hence, we get the desired result.
Here, we give an example of continuous $\mathcal{F}$-version of Ciric's quasi contraction map, which have a unique fixed point.

Example 3.5. Let $A = \{1,2,3\}$ with usual metric $\mathfrak{d}$, then $(A,\mathfrak{d})$ is a complete MS. Define a mapping $\mathfrak{T}:\Omega\to\Omega$ as $\mathfrak{T}(1) = 2, \mathfrak{T}(2) = 2, \mathfrak{T}(3) = 1$, which implies $\mathfrak{T}$ is continuous. Also define $\mathcal{F}:(0,\infty)\to\mathbb{R}$ such that $\mathcal{F}(t) = -\frac{1}{t}$. Then $\mathcal{F}$ fulfills both properties $(P1)$ and $(P2)$.
Let $\pi, b\in\Omega$, such that $\mathfrak{T}\pi\neq\mathfrak{T}b$. We have, $\mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi,\mathfrak{T}b)) = \mathcal{F}(1) = -1$, and

$$\max\{\mathfrak{d}(\pi,b),\mathfrak{d}(\pi,\mathfrak{T}a),\mathfrak{d}(b,\mathfrak{T}b),\mathfrak{d}(\pi,\mathfrak{T}b),\mathfrak{d}(b,\mathfrak{T}\pi)\} = 2.$$

Taking $\xi = \log 2$, then we get

$$\begin{aligned}
\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}\pi,\mathfrak{T}b)) &= \xi - 1 \\
&= \log 2 - 1 \\
&\le -\frac{1}{2} \\
&= -\frac{1}{\max\{\mathfrak{d}(\pi,b),\mathfrak{d}(\pi,\mathfrak{T}\pi),\mathfrak{d}(b,\mathfrak{T}b),\mathfrak{d}(\pi,\mathfrak{T}b),\mathfrak{d}(b,\mathfrak{T}\pi)\}} \\
&= \mathcal{F}(\max\{\mathfrak{d}(\pi,b),\mathfrak{d}(\pi,\mathfrak{T}\pi),\mathfrak{d}(b,\mathfrak{T}b),\mathfrak{d}(\pi,\mathfrak{T}b),\mathfrak{d}(b,\mathfrak{T}\pi)\})
\end{aligned}$$

We see that $\mathfrak{T}$ is continuous F-Ciric's quasi contraction and $\mathfrak{T}$ has a unique FP in Ω.
Now, we give positive answer to the open question, which is related to generalization of Sehgal's result.
Question 3.6. [Fabiano et.al. 2022, Question 4.2] Let $\mathfrak{T}$ be a self-map on a compact MS a $(\Omega,\mathfrak{d})$ satisfying
$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}^m\pi,\mathfrak{T}^m b)) \le \mathcal{F}(\mathfrak{d}(\pi,b)), \#(5)$
for all $\pi, b\in\Omega$ and for some $\xi > 0$ with $\mathfrak{T}^m\pi\neq\mathfrak{T}^m b$, also $\mathcal{F}$ fulfills conditions $(P1)$ and $(P2)$, and $m = m(\pi,b)$. Does there exist unique FP of $\mathfrak{T}$?

Proof. Since $\mathfrak{T}^m$ fulfills the conditions of Theorem 2.1. We attain that $\mathfrak{T}^m$ has a unique FP $\pi\in\Omega$.

Let $\mathrm{Fix}(\mathfrak{T}) = \{\pi\in\Omega: \mathfrak{T}\pi = \pi\}$. Suppose $Fix(\mathfrak{T}) = \emptyset$. Then there does not exists $\pi^*\in\Omega$ satisfying $\mathfrak{T}\pi^* = \pi^*$. This means $\mathfrak{d}(\pi^*,\mathfrak{T}\pi^*) > 0$ for all $\pi^*\in\Omega$. Then $\mathfrak{T}^m\pi^*\neq\mathfrak{T}^m\mathfrak{T}\pi^*$. If $\mathfrak{T}^m\pi^* = \mathfrak{T}^m\cdot\mathfrak{T}\pi^*$ then

$\mathfrak{T}(\mathfrak{T}^m\pi^*) = \mathfrak{T}^m\pi^*$, which implies $\mathfrak{T}^m\pi^*$ is a FP of $\mathfrak{T}$. Thus, $\mathfrak{T}^m\pi^* \neq \mathfrak{T}^m\mathfrak{T}\pi^*$. Then from contraction condition (5), we have,

$\xi + F(\mathfrak{d}(\mathfrak{T}^m\pi^*, \mathfrak{T}^m\mathfrak{T}\pi^*)) \leq F(\mathfrak{d}(\pi^*, \mathfrak{T}\pi^*))$.

Since $\mathfrak{T}$ fulfills $(P1)$ therefore we get

$\mathfrak{d}(\mathfrak{T}^m\pi^*, T^m\mathfrak{T}\pi^*)) < (\mathfrak{d}(\pi^*, \mathfrak{T}\pi^*)$.

This implies

$\mathfrak{d}(\pi^*, \mathfrak{T}\pi^*) < \mathfrak{d}(\pi^*, \mathfrak{T}\pi^*)$,

which gives a contradiction to the fact $\text{Fix}(\mathfrak{T}) = \emptyset$. Thus, there exists a FP of $\mathfrak{T}$. Also, we have, $\text{Fix}(\mathfrak{T}) \subseteq Fix(\mathfrak{T}^m)$. Hence, $\pi \in \Omega$ is a unique FP of $\mathfrak{T}$.

Remark 3.7. We have provided a proof of Question 3.6 using the property (P1) only, which is a weak condition for the mapping $\mathcal{F}$ that fulfils the condition $(P1)$ and $(P2)$.

Moreover, we see that the continuity condition of self-map $\mathfrak{T}$ in Question 3.6 is not assumed. Hence, we have a generalization of Sehgal's result.

Finally, we give an example of a map that fulfills the conditions of Question 3.6 having 0 as a unique FP of $\mathfrak{T}$.

Example 3.8. Let $\Omega = [0,1]$ with standard metric $\mathfrak{d}$. Then $(\Omega, \mathfrak{d})$ is a complete MS. Define mappings $\mathfrak{T}: \Omega \to \Omega$ as

$$\mathfrak{T}(a) = \begin{cases} \frac{\pi}{2} & \text{if } \pi \in [0,1) \\ 0 & \text{if } \pi = 1 \end{cases}$$

Take $\mathcal{F}: (0, \infty) \to \mathbb{R}$ such that $\mathcal{F}(t) = -\frac{1}{t}$.

Then $\mathcal{F}$ fulfills the property $(P1)$.

Case (i): Let $\pi, b \in [0,1)$. Then for $\mathfrak{T}^m\pi \neq \mathfrak{T}^m b$, we have

$$\mathcal{F}(\mathfrak{d}(\mathfrak{T}^m\pi, \mathfrak{T}^m b)) = \mathcal{F}\left(\left|\frac{\pi}{2^m} - \frac{b}{2^m}\right|\right)$$
$$= -\frac{2^m}{|\pi - b|}$$
$$= -\frac{2^m}{d(\pi, b)}.$$

Taking $\xi = \frac{1}{\mathfrak{d}(\pi,b)} > 0$, and $m = [\pi + b] + 1$ where [.] denotes greatest integer function. Then we get

$$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}^m\pi, T^m b)) \leq -\frac{1}{\mathfrak{d}(\pi, b)}$$
$$= \mathcal{F}(\mathfrak{d}(\pi, b))$$

Case (ii): Let $\pi \in [0,1), b = 1$. Clearly $\pi \neq 0$ otherwise we get $\mathfrak{T}^m\pi = \mathfrak{T}^m b$, then we have

$$\mathcal{F}(d(\mathfrak{T}^m\pi, \mathfrak{T}^m b)) = \mathcal{F}\left(\left|\frac{\pi}{2^m} - 0\right|\right)$$
$$= -\frac{2^m}{|\pi|}.$$

Taking $\xi = \frac{1}{|\pi|} > 0$ and $m = [\pi + b] + 2$, we get

$$\xi + \mathcal{F}(\mathfrak{d}(\mathfrak{T}^m\pi, \mathfrak{T}^m b)) \leq -\frac{1}{|\pi|}$$
$$\leq -\frac{1}{|\pi - b|}$$
$$= F(d(\pi, b))$$

Case (iii): Let $\pi = 0, b \in [0,1)$. Then this case is similar as Case (ii).

Case (iv): Let $\pi = 1, b = 1$. Then for all $n, \mathfrak{T}^m\pi = \mathfrak{T}^m b = 0$. Hence, $\mathfrak{T}$ fulfils all the conditions of Question 3.6.

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Conclusion

In this paper, we give a negative answer for the first problem [Fabiano et al. 2022, Question 4.1], but the answer is positive under certain assumptions [Fabiano et al. 2022, Question 4.1]. We get a positive answer for the second problem [Fabiano et.al 2022, Question 4.2], which is a generalization of Sehgal's result.

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