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Integer Solutions of Second Order Indeterminate Equations

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Abstract: This paper investigates second-order indeterminate equations, discuss the contributions of ancient Indian mathematicians to the field. The paper deals with the methods of Samāsabhāvana and the Chakravala algorithm for solving second-order indeterminate equations. This study provides a comprehensive analysis of Brahmagupta and Bhaskara's methods of solving these equations, highlighting the mathematical insights embedded in their works. The equation has various applications in modern algebra and number theory. We discuss thoroughly these methods and algorithm proofs in a straightforward manner, using examples to illustrate. The results obtained through these methods are presented in tabular form.

Keywords*:* Samāsabhāvana • Cakravāla • Brahmagupta • Bhaskara • Integral Solutions • Indeterminate Equation.

Introduction

The greatest accomplishments of ancient Indian mathematicians in number theory involved discovering integer solutions for Diophantine equations, which are equations with integer coefficients where the focus is on finding integer solutions (Dutta 2003). Brahmagupta (598 – 668 AD) was the first Indian mathematician who solved the second order indeterminate equation using a general method. Brahmagupta discovered a classical method to generate infinitely many integer solutions of an equation of the type $mx^2 + 1 = y^2$, where m is a positive and non-square integer (Arya 2014). For Example, if m = 2, then assuming $(x, y) = (2, 3)$ as the first solution of $2x^2 + 1 = y^2$ and generate the solutions successively: (12, 17), (77, 99), (408, 577) .... Similarly, if m = 3 then assuming $(x, y) = (1, 2)$ as a first solution of $3x^2 + 1 = y^2$ and generate the solutions successively: (4, 7), (15, 26), (56, 97) …. Brahmagupta was the first who proved that the integral solution of $mx^2 + 1 = y^2, m \neq$ a square integer depends on finding a solution of an auxiliary equation of the form $m{x_0}^2 + k = {y_0}^2$, $k = (\pm 1, \pm 2, \pm 4)$ and x_0, y_0 are positive integers. Then by repeated application of Brahmagupta's lemmas infinite number of solutions can be obtained (Arya 2014; Dutta et al., 1935; Srinivasiengar, 1967). Brahmagupta also made the discovery that in cases where the equation $mx^2 + k = y^2$ has a solution (a, b) with k being ±1, ±2, ±4, it is possible to find an integer solution to Pell's equation $mx^2 + k = y^2$. Although Brahmagupta consistently provided accurate solutions to Pell's equation, he was unable to obtain the general solution, rendering his method incomplete. His texts lack formal proofs, leaving us unaware of the mechanisms behind his approach.

Brahmagupta proposed the solution for the equation $mx^2 + 1 = y^2$ as: $x = \frac{2t}{t^2 - m}$ and $y = \frac{t^2 + m}{t^2 - m}$, where t can be substituted by any number. It can be verified in this manner as

$$mx^2+1=m\left(\frac{2t}{t^2-m}\right)^2+1=m\left(\frac{4t^2}{(t^2-m)^2}\right)+1=\frac{4mt^2+(t^2-m)^2}{(t^2-m)^2}=\frac{t^4+m^2-2mt^2+4mt^2}{(t^2-m)^2}=\frac{t^4+m^2+2mt^2}{(t^2-m)^2}=$$
$$\frac{(t^2+m)^2}{(t^2-m)^2}=\left(\frac{t^2+m}{t^2-m}\right)^2=y^2.$$

For example, if t = 3, the solution is $x=\frac{6}{9-m}$ and $y=\frac{9+m}{9-m}$, so that

$$mx^2+1=m\left(\frac{6}{9-m}\right)^2+1=\frac{36m}{81-18m+m^2}+1=\frac{36m+81-18m+m^2}{81-18m+m^2}=\frac{m^2+18m+81}{m^2-18m+81}=\left(\frac{9+m}{9-m}\right)^2=y^2$$
.

In the 10th century, Jayadeva (900-980 AD) used Chakravāla method to find integer solutions of the equations $mx^2+k=y^2$. Bhāskara II (1114 AD) also used Chakravāla method. Indian mathematicians used the term "*Varga-Prakriti*" or "*Kriti-Prakriti*" for the equation of the type $mx^2+k=y^2$ where N is integer. Varga means square and Prakriti means nature (Arya 2014). The term Prakriti used to refer N which is the coefficient of the square of the unknown. The term *Kshepa* or *Prakshepa* used to refer *k* quantity (Arya 2014; Dickson 1919).

In this equation of the form $mx^2+k=y^2$, if m is less than zero or m is a perfect square then infinitely many integral solutions can be easily obtained. Brahmagupta was only fascinated in solving more difficult equations where m is a positive integer but not a perfect square. Brahmagupta solved the following equations

$8x^2+1=y^2$: The solution is (x, y) = (1, 3), (6, 17), (35, 99), (204, 577), (1189, 3363) ……

$11x^2+1=y^2$: The solution is (x, y) = (3, 10), (161/5, 534/5) ……

$61x^2+1=y^2$: The smallest solution is x = 226153980 and y = 1766319049

Brahmagupta noted that that any person who is able to solve the equations $83x^2+1=y^2$ and $92x^2+1=y^2$ in a year will be a truly mathematician (Arya 2014). In 1657 AD, French mathematician Fermat who was studying these equations, challenged his contemporaries to solve the equation $61x^2+1=y^2$ but nobody succeeded in solving the equation in integers. Only in 1732 AD, Swiss mathematician, Leonard Euler solved the equation and incorrectly named this equation as Pell's Equation. English mathematician John Pell (1610-1685 AD) had just mentioned this equation. There is no contribution to this subject by Pell. Hence, Pell's equation must be named as Brahmagupta equation (Arya 2014; Dickson 1919; Puttaswamy 2012).

There is a notable achievement of Indian mathematicians that they could discover the methods to find integral solutions to indeterminate equations by 7th century. Aryabhata I was the first who achieved integral solutions for first order indeterminate equation (Puttaswamy 2012). In modern time, the indeterminate equations with integer coefficients are called as Diophantine Equations in the name of Greek mathematician Diophantus but Diophantus never focused on integer solutions. He only worked on rational solutions and never used general methods to solve equations. In the field of number theory, the most outstanding achievement accomplishments of ancient Indian mathematicians revolved around the discovery of integer solutions for Diophantine equations. These equations, which have integer coefficients, are referred to as 'Diophantine equations' when the objective is to identify their integer solutions.

2. Brahmagupta's Lemma or Samāsabhāvana (The Principal of Composition)

Brahmagupta gave an innovative composition law, by which a third solution can be obtained from the given two solutions of the equation. Brahmagupta's composition law shows how to combine two solutions of the equation $mx^2+k=y^2$, $m>0$. Here m and k are given integers, $m\neq$ a square integer. The problem is to find positive integer solutions (x, y) of the equation. There are some important lemmas of Brahmagupta which are basic of the discussion (Murthy, 1993)

2.1 Lemma 1 (Brahmagupta)
If $(x,y)=(\alpha,\beta)$ is a solution of the equation $mx^2+k=y^2$ and if $(x,y)=(\alpha',\beta')$ is a solution of the equation $mx^2+k'=y^2$ the

$$(x,y)=\begin{cases}(\alpha\beta'+\alpha'\beta,\ \beta\beta'+m\alpha\alpha')\\(\alpha\beta'-\alpha'\beta,\ \beta\beta'-m\alpha\alpha')\end{cases}$$

are the solutions of the equation $mx^2+kk'=y^2$.

Proof: Since $(x,y)=(\alpha,\beta)$ and $(x,y)=(\alpha',\beta')$ are the solutions of the equation $mx^2+k=y^2$ and $mx^2+k'=y^2$ respectively, therefore we have $m\alpha^2+k=\beta^2$ and $m\alpha'^2+k'=\beta'^2$. This implies that

$$(\beta^2-m\alpha^2)((\beta'^2-m\alpha'^2)=kk'.$$

Taking $(x,y)=(\alpha\beta'+\alpha'\beta,\ \beta\beta'+m\alpha\alpha')$, we see that

$$y^2-mx^2=(\beta\beta'+m\alpha\alpha')^2-m(\alpha\beta'+\alpha'\beta)^2=(\beta^2\beta'^2+m^2\alpha^2\alpha'^2+2m\beta\beta'\alpha\alpha')-m(\alpha^2\beta'^2+\alpha'^2+2\alpha\beta'\alpha'\beta)=\beta'^2(\beta^2-m\alpha^2)-m\alpha'^2(\beta^2-m\alpha^2)$$

$$=(\beta^2-m\alpha^2)((\beta'^2-m\alpha'^2)=kk'.$$

Hence, $y^2-mx^2=kk'\Rightarrow mx^2+kk'=y^2$.

Thus, (x, y) = $(\alpha\beta'+\alpha'\beta,\ \beta\beta'+m\alpha\alpha')$ is a solution of the equation $mx^2+kk'=y^2$. Hence proved.

Next, we shall show that (x, y) = $(\alpha\beta'-\alpha'\beta,\beta\beta'-m\alpha\alpha')$ is a solution of the equation $mx^2+kk'=y^2$. We have,

$$y^2-mx^2=(\beta\beta'-m\alpha\alpha')^2-m(\alpha\beta'-\alpha'\beta)^2=(\beta^2\beta'^2+m^2\alpha^2\alpha'^2-2m\beta\beta'\alpha\alpha')-m(\alpha^2\beta'^2+\alpha'^2-2\alpha\beta'\alpha'\beta)=\beta'^2(\beta^2-m\alpha^2)-m\alpha'^2(\beta^2-m\alpha^2)-2m\beta\beta'\alpha\alpha'+2m\alpha\beta'\alpha'\beta=(\beta^2-m\alpha^2)(\beta'^2-m\alpha'^2)$$

$$=kk'$$

Hence, $y^2-mx^2=kk'\Rightarrow mx^2+kk'=y^2$.

Thus, (x, y) = $(\alpha\beta'-\alpha'\beta,\ \beta\beta'-m\alpha\alpha')$ is also a solution of the equation $mx^2+kk'=y^2$. Hence proved.

2.2 Lemma 2 (Brahmagupta)
If $(x,y)=(\alpha,\beta)$ is a solution of the equation $mx^2+k=y^2$ then $(x,y)=(2\alpha\beta,\beta^2+m\alpha^2)$ is a solution of the equation $mx^2+k^2=y^2$.

Proof: This follows from lemma 1 when we take $k=k'and\ (\alpha',\beta')=(\alpha,\beta)$. It can be directly verified that

$$y^2-mx^2=(\beta\beta'+m\alpha\alpha')^2-m(\alpha\beta'+\alpha'\beta)^2$$
$$=(\beta^2+m\alpha^2)^2-m(\alpha\beta+\alpha\beta)^2,$$

since $(\alpha',\beta')=(\alpha,\beta)$

$$=(\beta^2+m\alpha^2)^2-m(2\alpha\beta)^2$$
$$=(\beta^2-m\alpha^2)^2$$
$$=k^2$$

, since $k=\beta^2-m\alpha^2$ because $(x,y)=(\alpha,\beta)$ is a solution of the equation $mx^2+k=y^2$. Proved

2.3 Lemma 3 (Brahmagupta)
If $(x,y)=(\alpha,\beta)$ is a solution of the equation $mx^2+k^2=y^2$ such that k divides α and k divides β then $(x,y)=\left(\frac{\alpha}{|k|},\frac{\beta}{|k|}\right)$ is a solution of the equation $mx^2+1=y^2$.

Proof: Since $\alpha'=\frac{\alpha}{k}$ and $\beta'=\frac{\beta}{k}$ are integers, therefore we have

$$\beta'^2-m\alpha'^2=\left(\frac{\beta}{|k|}\right)^2-m\left(\frac{\alpha}{|k|}\right)^2$$

$$\Rightarrow\beta'^2-m\alpha'^2=\frac{\beta^2}{k^2}-m.\frac{\alpha^2}{k^2}$$

$(\because\ |k|^2=k^2)$

$$\Rightarrow\ \beta'^2-m\alpha'^2=\frac{1}{k^2}(\beta^2-m\alpha^2)$$

… (1)

Since $(x,y)=(\alpha,\beta)$ is a solution of the equation $mx^2+k^2=y^2$ then we have $m\alpha^2+k^2=\beta^2\Rightarrow k^2=\beta^2-m\alpha^2$. Now, putting this value of k^2 in equation (1). Then

$$\beta'^2-m\alpha'^2=\frac{1}{k^2}(\beta^2-m\alpha^2)=\frac{1}{k^2}.k^2=1$$

$$\Rightarrow\ m\alpha'^2+1=\beta'^2\Rightarrow\ (x,y)=(\alpha',\beta')=\left(\frac{\alpha}{|k|},\frac{\beta}{|k|}\right)$$

is a solution of the equation $mx^2+1=y^2$.

Brahmagupta was the first mathematician who systematically studied the problem of finding infinitely many integral solutions of second order indeterminate equations. His

Samāsabhāvana is a special technique devised with fundamental concept of modern algebra. It is amazing that Brahmagupta displayed an advanced algebra thinking during 7th century whereas algebra was still in infancy in Arab and Europe. Arabs and European were struggling to solve first order indeterminate equations till 16th century. Only in 1758 AD, Euler rediscovered Brahmagupta's Lemma.

3. Important Applications of Brahmagupta's Lemma

Brahmagupta's Lemma gives the establishment of algebraic structure for the integral solutions of the equations $mx^2 + k = y^2$, where m is a positive integer. For sake of convenience of the discussion, we shall denote a triple of integers that satisfy $mx^2 + k_1 = y^2$ and $mx^2 + k_2 = y^2$ as $(x_1, y_1; k_1)$ $(x_2, y_2; k_2)$ and Brahmagupta's Lemma 1 introduces two binary composition laws denoted $\otimes$ by :

$(x_1, y_1; k_1)\otimes(x_2, y_2; k_2) = (x_1y_2 + x_2y_1,\ y_1y_2 + mx_1x_2;\ k_1k_2)$

&

$(x_1,y_1;k_1)\otimes(x_2, y_2; k_2) = (x_1y_2 - x_2y_1,\ y_1y_2 - mx_1x_2;\ k_1k_2)$

. These laws are commonly referred to bhāvana (the principle of composition) rules, where the positive sign corresponding to the samāsabhāvana (additive composition) and the negative sign corresponding to the antarabhāvana (subtractive composition) (Colebrooke, 1817). Therefore, Brahmagupta's Lemma provides composition laws specifically for the set of integral solutions of the equation $mx^2 + 1 = y^2$. The samāsabhāvana is possibly the earliest example of abstract algebraic thinking. The samāsabhāvana allows one to generate infinitely many integral solutions to the equation $mx^2 + 1 = y^2$ from a given non-trivial integral solution. If x, y are positive integers that satisfy $mx^2 + 1 = y^2$, we define $(x_0, y_0) = (x_1, y_1)$ and $(x_{i+1}, y_{i+1}) = (x_1, y_1) \otimes (x_i, y_i)$, where $\otimes$ represents the samāsabhāvana. It is evident that $x_{i+1} > x_i > ... > x_1 > x_0$ and $y_{i+1} > y_i > ... > y_1 > y_0$, indicating that the solutions are increasing and distinct (Dutta, 2003). Thus, everyone can generate infinitely many solutions.

Consider a second order indeterminate equation

$$mx^2 + 1 = y^2$$

$$\therefore m = \frac{(y^2-1)}{x^2}$$

$$\Rightarrow m = \frac{y^2}{x^2} - \frac{1}{x^2}$$

If $x \to \infty$

$$\Rightarrow \sqrt{m} \approx \frac{y}{x} \quad \dots (2)$$

From above, in search of finding integer solution of $mx^2 + 1 = y^2$, we conclude that if $x \to \infty$, then $y \to \infty$ because $y^2 - mx^2 = 1$, where m > 0. Thus, for a sufficiently large integer solution (x, y) of the equation $mx^2 + 1 = y^2$, $\frac{y}{x}$ will be a good approximation for $\sqrt{m}$.

Let us initially examine a few consequences of Brahmagupta's Lemma in relation to the equation. Before discussing further, some important application of Brahmagupta's Lemma or Samāsabhāvana for m = 83 and m = 92 are provided in the following sections 3.1 and 3.2 respectively.

3.1 Application of Brahmagupta's Lemma in Case of m = 83

Finding rational solutions of the equation $83x^2 + 1 = y^2$ is very simple other than integer solutions because all rational solutions can be obtained by given Brahmagupta's method for the equation $mx^2 + 1 = y^2$ as: $x = \frac{2t}{t^2-m}$ and $y = \frac{t^2+m}{t^2-m}$, where t can be replaced by integer. The real interest lies in the search for positive integer solutions of the equation $83x^2 + 1 = y^2$. We can find integer solution of the given equation $mx^2 + 1 = y^2$ by Brahmagupta's Samāsabhāvana. First, we try to find an integer solution (x_0, y_0) of an

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auxiliary equation $mx^2 + k = y^2$ by trial, where we have to choose value of integer k such that $mx_0{}^2 + k = y_0{}^2$. Thus, we can find an initial solution (x_0, y_0). Then, by using Brahmagupta's Lemmas, we can find infinitely many integers solution of the given equation. But here given that m = 83, so auxiliary equation for $83x^2 + 1 = y^2$ is $83x^2 + k = y^2$. Taking value of $x = x_0 = 1$ and $k = -2$, we have $83x_0{}^2 - 2 = 83 - 2 = 81 = y_0{}^2$. This implies that $y_0 = 9$. Thus, $(x_0, y_0) = (1,9)$ is a solution of auxiliary equation $83x^2 - 2 = y^2$.

By lemma 2, $(x_1, y_1) = (2x_0y_0,\ mx_0{}^2 + y_0{}^2) = (18, 164)$ is a solution of the equation $83x^2 + (-2)^2 = y^2$.

By lemma 3, $(x_2, y_2) = \left(\frac{x_1}{|-2|}, \frac{y_1}{|-2|}\right) = \left(\frac{18}{2}, \frac{164}{2}\right) = (9, 82)$ is a solution of the equation $83x^2 + 1 = y^2$. Hence, $(x_2, y_2) = (9, 82)$ is a solution of $83x^2 + 1 = y^2$. Using Lemma 1 or lemma 2 between (x_2, y_2) and (x_2, y_2) itself, we get a third solution

$(x_3, y_3) = (x_2y_2 + x_2y_2, y_2y_2 + 83x_2x_2) = (2x_2y_2, 83x_2{}^2 + y_2{}^2) = (1476, 13447)$.

Again, by applying Brahmagupta's lemma 2 between (x_3, y_3) itself, we get another solution (x_4, y_4). Then by repeated application of Brahmagupta's lemmas any number of solutions can be obtained.

3.2 Application of Brahmagupta's Lemma in Case of m = 92

The real interest lies in the search for positive integer solutions of the equation $92x^2 + 1 = y^2$. Instead look at auxiliary equation $92x^2 + k = y^2$. Let us consider initial solution $(x_0, y_0) = (1,10)$ and k = 8 such that $92x^2 + k = y^2$ satisfied. Thus, $(x_0, y_0) = (1,10)$ is a solution of auxiliary equation $92x^2 + 8 = y^2$, where m = 92 and k= 8

By lemma 2, $(x_1, y_1) = (2x_0y_0,\ mx_0{}^2 + y_0{}^2) = (20, 192)$ is a solution of the equation $92x^2 + 8^2 = y^2$

By lemma 3, $(x_2, y_2) = \left(\frac{x_1}{8}, \frac{y_1}{8}\right) = \left(\frac{20}{8}, \frac{192}{8}\right) = \left(\frac{5}{2}, 24\right)$ is a solution of the equation $92x^2 + 1 = y^2$.

But the solution $(x_2, y_2) = \left(\frac{5}{2}, 24\right)$ is a rational solution, not integer solution. Next, we proceed to find integer solutions. Since, $(x_2, y_2) = \left(\frac{5}{2}, 24\right)$ is a solution of the equation $92x^2 + 1 = y^2$, where m = 92 and k = 1 therefore by using lemma 2 between (x_2, y_2) itself, we get a third solution $(x_3, y_3) = (x_2y_2 + x_2y_2, y_2y_2 + 92x_2x_2) = (2x_2y_2, 92x_2{}^2 + y_2{}^2) = (120, 1151)$ is an integer solution for the equation $92x^2 + 1 = y^2$.

Then by repeated application of Brahmagupta's lemmas, any number of solutions of the equation $92x^2 + 1 = y^2$ in the form

$(x_{i+1}, y_{i+1}) = (2x_iy_i, 92x_i{}^2 + y_i{}^2), i = 2,3,4, \ldots$

can be obtained such that $x_i \to \infty$ and $y_i \to \infty$ as shown in the table 1.

Numerical results for $92x^2 + 1 = y^2$		
x_i, i = 0,1,2,…	y_i, i = 0,1, 2,…	$(x_{i+1}, y_{i+1}) = (2x_iy_i, 92x_i{}^2 + y_i{}^2)$
1	10	(20, 192)
5/2	24	(120, 1151)
120	1151	(276240, 2649601)
276240	2649601	(1463851560480, 14040770918401)
1463851560480	14040770918401	(41107208838487013192784960, 394286495966030522000793601)

Table 1.

3.3 Approximate Value of $\sqrt{2}$ by Second Order Indeterminate Equation

Let us consider a second order indeterminate equation $2x^2 + 1 = y^2$. First, we have to solve this equation and then approximation of $\sqrt{2}$ has been given. The real interest lies in the search for positive integer solutions of the equation $y^2 - 2x^2 = 1$. By inspection, initial solution $(x_0, y_0) =$ (2, 3) of $2x^2 + 1 = y^2$, where m = 2.

By lemma 2, $(x_1, y_1) = (2x_0y_0,\ mx_0^2 + y_0^2) = (12, 17)$ be a solution of $2x^2 + 1 = y^2$. Similarly, by using lemma 1 between (x_1, y_1) and (x_0, y_0) then, we get $(x_2, y_2) = (x_1y_0 + x_0y_1, mx_0x_1 + y_0y_1) = (70, 99)$ as a solution of $2x^2 + 1 = y^2$. Again, using lemma 1 between (x_2, y_2) and (x_0, y_0), we have obtained next solution of $2x^2 + 1 = y^2$ as

$(x_3, y_3) = (x_2y_0 + x_0y_2, mx_0x_2 + y_0y_2) = (408, 577)$

In the search of positive integer solutions (x, y) of $mx^2 + 1 = y^2$ as $x \to \infty$ then $y \to \infty$ and $\sqrt{m} \approx \frac{y}{x}$, by equation (2). Hence,

In the search of positive integer solutions (x, y) of $2x^2 + 1 = y^2$ as $x \to \infty$ then $y \to \infty$ and $\sqrt{2} \approx \frac{y}{x}$, by equation (2).

Therefore

$$\sqrt{2} \approx \frac{y_3}{x_3} \Rightarrow \sqrt{2} \approx \frac{577}{408} = 1.41421569$$

Since Boudhayan's used

$$\sqrt{2} \approx 1 + \frac{1}{3} + \left(\frac{1}{4} \times \frac{1}{3}\right) - \left(\frac{1}{34} \times \frac{1}{4} \times \frac{1}{3}\right) \Rightarrow \sqrt{2} \approx \frac{577}{408}$$

(Dutta 1932).

Hence, we conclude that separated by more than 1000 years, Brahmagupta and Boudhayana come out with the same approximation for $\sqrt{2}$.

In the same manner,

If (x_i, y_i) is the first integer solution of the corresponding second order indeterminate equation $x^2 - Dy^2 = 1$. The rational approximation of corresponding irrational number $\sqrt{D}$ is $\frac{x_i}{y_i}$.

Examples:

a. The approximation of $\sqrt{58}$ in the cakravāla process are:

$$\sqrt{58} \approx \frac{x_1}{y_1} = \frac{8}{1}, \frac{x_2}{y_2} = \frac{23}{3}, \frac{x_3}{y_3} = \frac{38}{5}, \frac{x_4}{y_4} = \frac{99}{13}, \frac{x_5}{y_5} = \frac{19603}{2574} = 7.6157731157...$$

. This value corrects to 7 places of decimals to the actual value.

b. The approximation of $\sqrt{61}$ in the cakravāla process are:

$$\sqrt{61} \approx \frac{x_1}{y_1} = \frac{8}{1}, \frac{x_2}{y_2} = \frac{39}{5}, \frac{x_3}{y_3} = \frac{164}{21}, \dots, \frac{x_{14}}{y_{14}} = \frac{1766319049}{226153980} = 7.81024967590665439538...$$

. This value corrects to 17 places of decimals to the actual value.

c. The approximation of $\sqrt{97}$ in the cakravāla process are:

$$\sqrt{97} \approx \frac{x_1}{y_1} = \frac{10}{1}, \frac{x_2}{y_2} = \frac{69}{7}, \frac{x_3}{y_3} = \frac{197}{20}, \dots, \frac{x_{12}}{y_{12}} = \frac{62809633}{6377352} = 9.848857801796105...$$

. This value corrects to 14 places of decimals to the actual value.

4. Cakravāla Method of Bhaskara

Brahmagupta did not have a method for deriving an auxiliary equation in the specified way. However, it was Bhaskara II who introduced the brilliant and effective Cakravāla, or cyclic method, which provided an explanation for constructing such an auxiliary equation along with its two integer solutions. There are some important lemmas and method of Bhaskara which are basic of the discussion (Murthy, 1993).

4.1 Lemma 4 (Bhaskara)

If $F(x, y) = Dx^2 - y^2$ and $h_0 = F(x_0, y_0)$ and

$$(x_1, y_1, h_1) = \left(\frac{m_0x_0 + y_0}{h_0}, \frac{Dx_0 + m_0y_0}{h_0}, \frac{D - m_0^2}{h_0}\right) \quad \dots (3)$$

then one has the identity

$$Dx_1^2 - y_1^2 = -h_1 \quad \dots (4)$$

i.e.,

$$F(x_1, y_1) = -h_1 \quad \dots (5)$$

We shall refer to (4) as Bhaskara's identity.

Proof: We have

$$F(x_1, y_1) = Dx_1^2 - y_1^2 = D\left(\frac{m_0x_0 + y_0}{h_0}\right)^2 - \left(\frac{Dx_0 + m_0y_0}{h_0}\right)^2$$

$$= \frac{1}{h_0^2}\left[D(m_0^2x_0^2 + 2m_0x_0y_0 + y_0^2) - (D^2x_0^2 + 2Dm_0x_0y_0 + m_0^2y_0^2)\right]$$

$= \frac{1}{h_0{}^2}[Dx_0{}^2(m_0{}^2 - D) - y_0{}^2(m_0{}^2 - D)]$

$= \frac{1}{h_0{}^2}[(Dx_0{}^2 - y_0{}^2)(m_0{}^2 - D)] \quad = \frac{1}{h_0{}^2}h_0(m_0{}^2 - D) = -\frac{D-m_0{}^2}{h_0} = -h_1$

Let us assume that x_0, y_0 are coprime i.e., $gcd(x_0, y_0) = 1$. Since, $h_0 = F(x_0, y_0) = Dx_0{}^2 - y_0{}^2$, it is clear that x_0, y_0 and h_0 are mutually coprime. If we determine the positive integer m_0 such that

$m_0 x_0 + y_0 \equiv 0 (mod\ h_0)$ … (6)

Then $x_1 = \frac{m_0x_0+y_0}{h_0}$ is an integer. We claim that y_1 *and* h_1 in (3) are also integers. Since $h_0 = Dx_0{}^2 - y_0{}^2$, we have

$Dx_0{}^2 \equiv y_0{}^2(mod\ h_0)$

By (6), $m_0x_0 \equiv - y_0(mod\ h_0)$

Multiplying the latter congruence by y_0 and adding to the preview one, we get

$Dx_0{}^2 + m_0x_0y_0 \equiv 0(mod\ h_0)$

$\Rightarrow x_0(Dx_0 + m_0y_0) \equiv 0(mod\ h_0)$

Since $(x_0, h_0) = 1$, this implies that

$(Dx_0 + m_0y_0) \equiv 0(mod\ h_0)$ … (7)

i.e., $y_1 = \frac{Dx_0+m_0y_0}{h_0}$ is an integer.

Again, from (6) , $y_0 \equiv - m_0x_0(mod\ h_0)$; substituting this in (7) we get $(Dx_0 - m_0{}^2x_0) \equiv 0(mod\ h_0) \Rightarrow x_0(D - m_0{}^2) \equiv 0(mod\ h_0)$

Since, $(x_0, h_0) = 1$, this implies that

$(D - m_0{}^2) \equiv 0(mod\ h_0)$ … (8)

i.e., $h_1 = \frac{D-m_0{}^2}{h_0}$ is an integer.

Next, we observe that

$x_0y_1 - x_1y_0 = x_0\left(\frac{Dx_0+m_0y_0}{h_0}\right) - \left(\frac{m_0x_0+y_0}{h_0}\right)y_0 = \frac{1}{h_0}(Dx_0{}^2 - y_0{}^2) = 1$

i.e.,

$x_0y_1 - x_1y_0 = 1$

Consequently, x_1, y_1, h_1 must be mutually coprime.

4.2 Bhaskara's method of solving the equation $\boldsymbol{Dx^2 + 1 = y^2}$ ($\boldsymbol{D > 0, D \neq}$ a square integer)

Let $F(x, y) = Dx^2 - y^2$. On putting $x_0 = 1, y_0 = [\sqrt{D}], h_0 = F(x_0, y_0)$, where $h_0 > 0$ … (10)

If we put, as in (3), we get

$(x_1, y_1, h_1) = \left(\frac{m_0x_0+y_0}{h_0}, \frac{Dx_0+m_0y_0}{h_0}, \frac{D-m_0{}^2}{h_0}\right)$

Then, by Bhaskara's identity

$F(x_1, y_1) = -h_1$
$(h_1 > 0)$
… (11)

Here, m_0 must satisfy the congruence

$m_0x_0 + y_0 \equiv 0\ (mod\ h_0)$ … (12)

We now prescribed that the multiplier m_0 must satisfy the inequalities

$m_0 < \sqrt{D} < m_0 + h_0$ … (13)

Similarly, we can proceed with (x_1, y_1, h_1) as we did with (x_0, y_0, h_0). In this same manner, we can define inductively, the four sequences of positive integers $(m_\lambda, x_\lambda, y_\lambda, h_\lambda)$ $(\lambda = 0,1,2,3, \ldots)$ which satisfy the properties

$h_\lambda(-1)^\lambda = F(x_\lambda, y_\lambda)$,

$(x_{\lambda+1}, y_{\lambda+1}, h_{\lambda+1}) = \left(\frac{m_\lambda x_\lambda+y_\lambda}{h_\lambda}, \frac{Dx_\lambda+m_\lambda y_\lambda}{h_\lambda}, \frac{D-m_\lambda{}^2}{h_\lambda}\right)$, … (14)

$m_\lambda x_\lambda + y_\lambda \equiv 0\ (mod\ h_\lambda)$

$m_\lambda < \sqrt{D} < m_\lambda + h_\lambda$

Then, the claim of Bhaskara's theorem is that $\exists\ \lambda$ such that $h_\lambda = 1 \Rightarrow F(x_\lambda, y_\lambda) = (-1)^\lambda$ i.e., (x_λ, y_λ) is a solution of the equation $Dx^2 - y^2 = (-1)^\lambda$. If λ is odd, this means that we can solve $Dx^2 + 1 = y^2$. If λ is even, we can solve $Dx^2 - 1 = y^2$. If we obtain $h_\lambda = \pm 2, \pm 4$, then we can use Brahmagupta's lemma to complete the solution.

Example 1. What is that number whose square, multiplied by 75 and then added to unity, yields a perfect square?

We have to solve $Dx^2 + 1 = y^2, D = 75$. We have $F(x, y) = Dx^2 - y^2$.

Since $8^2 < D < 9^2 \Rightarrow 8 < \sqrt{D} < 9$, we have $[\sqrt{D}] = 8$. Thus $[\sqrt{D}] = 8 = y_0$. Taking $x_0 = 1$ by equation (10), we have $(x_0, y_0) = (1, 8)$. Then

$h_0 = F(x_0, y_0) = Dx_0{}^2 - y_0{}^2 = 75 - 64 = 11$.

Thus, $(x_0, y_0, h_0) = (1, 8, 11)$

In order to find next iteration (x_1, y_1, h_1), we have to solve the congruence

$$mx_0 + y_0 \equiv 0 \ (mod\ h_0)$$

$\Rightarrow m + 8 \equiv 0 \ (mod\ 11)$

$\Rightarrow m \equiv -8 \ (mod\ 11) \Rightarrow m \equiv 3 \ (mod\ 11)$

Solution of the above congruence in arithmetic progression is given by

$$m = 3, 14, 25, ...$$

We have $3 < \sqrt{D} < 14$. Hence $m_0 = 3$. Then

$$x_1 = \frac{m_0x_0 + y_0}{h_0} = \frac{3\times1+8}{11} = \frac{11}{11} = 1$$

$$y_1 = \frac{Dx_0 + m_0y_0}{h_0} = \frac{75\times1+3\times8}{11} = \frac{99}{11} = 9$$

$$h_1 = \frac{D - m_0{}^2}{h_0} = \frac{75-9}{11} = \frac{66}{11} = 6$$

Thus,

$(x_1, y_1, h_1) = (1, 9, 6)$

In order to find next iteration (x_2, y_2, h_2), we have to solve the congruence

$mx_1 + y_1 \equiv 0 \ (mod\ h_1)$

$\Rightarrow m + 9 \equiv 0 \ (mod\ 6) \Rightarrow m \equiv -9 \ (mod\ 6)$

$\Rightarrow m \equiv 3 \ (mod\ 6)$

Solution of the above congruence in arithmetic progression is given by

$$m = 3, 9, 15, ...$$

We have $3 < \sqrt{D} < 9$

Hence $m_1 = 3$. Then,

$$x_2 = \frac{m_1x_1 + y_1}{h_1} = \frac{3\times1+9}{6} = \frac{12}{6} = 2$$

$$y_2 = \frac{Dx_1 + m_1y_1}{h_1} = \frac{75\times1+3\times9}{6} = \frac{102}{6} = 17$$

$$h_2 = \frac{D - m_1{}^2}{h_1} = \frac{75-9}{6} = \frac{66}{6} = 11.$$

Hence,

$(x_2, y_2, h_2) = (2, 7, 11).$

Now, solving the congruence

$mx_2 + y_2 \equiv 0 \ (mod\ h_2)$

$\Rightarrow 2m + 17 \equiv 0 \ (mod\ 11)$

$\Rightarrow 2m \equiv -17 \ (mod\ 11)$

$\Rightarrow 2m \equiv 5 \ (mod\ 11)$

Solution of the above congruence in arithmetic progression is given by

$$m = 8, 19, 30, ...$$

We have, $8 < \sqrt{D} < 19$, thus $m_2 = 8$. Then

$$x_3 = \frac{m_2x_2 + y_2}{h_2} = \frac{8\times2+17}{11} = \frac{33}{11} = 3$$

$$y_3 = \frac{Dx_2 + m_2y_2}{h_2} = \frac{75\times2+8\times17}{11} = \frac{150+136}{11} = \frac{286}{11} = 26$$

$$h_3 = \frac{D - m_2{}^2}{h_2} = \frac{75-64}{11} = \frac{11}{11} = 1$$

Hence, we arrived at a stage where $h_\lambda = 1$, for some $\lambda = 3$. Thus, by the claim of Bhaskara's theorem, $(x_3, y_3) = (3, 26)$ is a solution of $75x^2 + 1 = y^2$. Since we have obtained an integer solution then by using repeated application of Brahmagupta's lemmas, we can easily find infinitely many solutions.

Example 2. What is that number whose square, multiplied by 61 and then added to unity, yields a perfect square?

We have to solve $Dx^2 + 1 = y^2$, $D = 61$. We have $F(x, y) = Dx^2 - y^2$.

Since $7 < \sqrt{61} < 8$, we take $(x_0, y_0) = (1, 7)$. Then $h_0{}^2 = Dx_0{}^2 - y_0{}^2 = 61 - 49 = 12.$ Solving the congruence

$$mx_0 + y_0 \equiv 0 \ (mod\ h_0)$$

$$\Rightarrow m + 7 \equiv 0 \ (mod\ 12)$$

$$\Rightarrow m \equiv 5 \ (mod\ 12)$$

Solution of the above congruence in arithmetic progression is given by

$$m = 5, 17, 29, ...$$

We have $5 < \sqrt{D} < 17$. Hence $m_0 = 3$. Hence, $(m_0, x_0, y_0, h_0) = (5, 1, 7, 12)$. We find

$$x_1 = \frac{m_0x_0 + y_0}{h_0} = \frac{5\times1+7}{12} = \frac{12}{12} = 1$$

$$y_1 = \frac{Dx_0 + m_0y_0}{h_0} = \frac{61\times1+5\times7}{12} = \frac{96}{12} = 8$$

$$h_1 = \frac{D - m_0{}^2}{h_0} = \frac{61-25}{12} = \frac{36}{12} = 3$$

Hence,

$$(x_1, y_1, h_1) = (1, 8, 3)$$

Next, we solve the congruence

$$mx_1 + y_1 \equiv 0 \ (mod\ h_1)$$

$$\Rightarrow m + 8 \equiv 0 \ (mod\ 3)$$

$$\Rightarrow m \equiv -8 \ (mod\ 3)$$

$$\Rightarrow m \equiv 1 \ (mod\ 1)$$

In the sequence $m = 1, 4, 7, 10, ...$ we must take $m_1 = 7$ because $7 < \sqrt{D} < 10$. We then find $(m_1, x_1, y_1, h_1) = (7, 1, 8, 3)$ and

$$x_2 = \frac{m_1x_1 + y_1}{h_1} = \frac{7\times1+8}{3} = \frac{15}{3} = 5$$

$$y_2 = \frac{Dx_1+m_1y_1}{h_1} = \frac{61\times1+7\times8}{3} = \frac{117}{3} = 39$$
$$h_2 = \frac{D-m_1{}^2}{h_1} = \frac{61-49}{3} = \frac{12}{3} = 4.$$

Since $h_\lambda(-1)^\lambda = F(x_\lambda, y_\lambda)$, and $h_2 = 4$, therefore $h_2 = F(x_2, y_2) \Rightarrow Dx_2{}^2 - y_2{}^2 = 4$ with $(x_2, y_2) = (5, 39)$. By Brahmagupta's lemma, $(\alpha,\beta) = \left(\frac{x_2}{2}, \frac{y_2}{2}\right) = \left(\frac{5}{2}, \frac{39}{2}\right)$ satisfies the equation

$$D\alpha^2 - 1 = \beta^2$$ with $(\alpha,\beta) = \left(\frac{5}{2}, \frac{39}{2}\right)$.

Composing the solution (α,β) with itself by Brahmagupta's lemma, we find a rational solution not an integer solution

$$(\alpha_1,\beta_1) = (2\alpha\beta, D\alpha^2+\beta^2) = \left(\frac{195}{2}, \frac{1523}{2}\right)$$

must satisfy the equation

$$D\alpha_1{}^2 + 1 = \beta_1{}^2, (\alpha_1,\beta_1) = \left(\frac{195}{2}, \frac{1523}{2}\right)$$

Applying Brahmagupta's lemma between (α,β) and (α_1,β_1), we get finally an integer solution which is required $(\xi,\eta) = (\alpha\beta_1 + \alpha_1\beta, D\alpha\alpha_1 + \beta\beta_1) = (3805, 29718)$ must satisfy the relation

$$D\xi^2 - 1 = \eta^2, \quad (\xi,\eta) = (3805, 29718)$$

Since once an integer solution obtained then by repeated application of Brahmagupta's lemma, we can get infinitely many solutions.

Example 3. The equation $Dx^2 + 1 = y^2$ with $D = 103$, $F(x,y) = Dx^2 - y^2$.

First sequence of multipliers $\{m_\lambda\}$ is determined successively, along with the sequences $\{x_\lambda\}, \{y_\lambda\}, \{h_\lambda\}$. The aim is to arrive at λ_0 with $h_{\lambda_0} = 1$. Also, as soon as we arrive at λ' with $h_{\lambda'} = \pm 1, \pm 2, \pm 4$, we can also use Brahmagupta's lemmas to complete the solution.

Accordingly, with $D = 103$, we have

$$10 < \sqrt{103} < 11, (x_0, y_0, h_0) = (1, 10, 3)$$

Then we find m_0 with

$$\begin{cases} m_0x_0 + y_0 \equiv 0 \ (mod\ h_0) \\ m_0 < \sqrt{D} < m_0 + h_0 \end{cases}$$

and we get $m_0 = 8$, so that

$$(x_1, y_1, h_1) = (6, 61, 13)$$

And so on.

We shall not present all the details of the computations, but display the obtained results in the following table.

Table 2:

Numerical Results for $D = 103$												
λ	0	1	2	3	4	5	6	7	8	9	10	11
m_λ	8	5	7	2	9	9	2	7	5	8	10	10
x_λ	1	6	7	20	27	47	450	497	947	2391	3338	**22419**
y_λ	10	61	71	203	274	477	4567	5044	9611	24266	33877	**227528**
h_λ	3	13	6	9	11	2	11	9	6	13	3	**1**

In table 2, we see that $h_{11} = 2$ and $2 \in \{\pm 1, \pm 2, \pm 4\}$. Here, we can use Brahmagupta's lemmas to find integer solutions of $103x^2 + 1 = y^2$. But here we tried to find $h_\lambda = 1$ by further calculations. Here, we conclude that $h_\lambda = 1$, for some $\lambda = 11$ and $(x_{11}, y_{11}) = (22419, 227528)$. The smallest solution $x = 22419, y = 227528$ comes after the 11th step. By claim of Bhaskara

$$h_\lambda = 1 \Rightarrow F(x_\lambda, y_\lambda) = (-1)^\lambda \Rightarrow F(x_{11}, y_{11}) = (-1)^{11}, \text{ for } \lambda = 11$$
$$\Rightarrow Dx_{11}{}^2 - y_{11}{}^2 = -1 \Rightarrow Dx_{11}{}^2 + 1 = y_{11}{}^2, \text{ where } D = 103$$

Hence, $(x_{11}, y_{11}) = (22419, 227528)$ satisfies $103x^2 + 1 = y^2$. Further, we can obtain infinitely many solutions by repetitive use of Brahmagupta's lemmas.

Example 4. The equation $Dx^2 + 1 = y^2$ with $D = 97$

In table 3, we have $h_{10} = 1$, therefore $(x_{10}, y_{10}) = (569, 5604)$ satisfies $97x^2 + 1 = y^2$, i.e., $D{x_{10}}^2 + 1 = {y_{10}}^2$. Hence, by applying Brahmagupta's lemma between (x_{10}, y_{10}) itself, we get other solution of $97x^2 + 1 = y^2$ as

$(\alpha, \beta) = (2x_{10}y_{10}, 97{x_{10}}^2 + {y_{10}}^2) = (6377352, 62809633)$ must satisfy $97x^2 + 1 = y^2$.

In this manner, by repeated application Brahmagupta's lemma, we can get infinitely many solutions.

Numerical Results for $D = 97$											
λ	0	1	2	3	4	5	6	7	8	9	10
m_λ	7	8	3	5	4	5	3	8	7	9	9
x_λ	1	1	6	7	13	20	33	53	86	483	**569**
y_λ	9	10	59	69	128	197	325	522	847	4757	**5604**
h_λ	16	3	11	8	9	9	8	11	3	16	**1**

Table 3

4.3 The process of Jayadeva and Bhāskara II

To solve $x^2 - Dy^2 = 1$, we have considered equations of the form $x_i^2 - Dy_i^2 = k_i$ in the intermediate stages. Here x_i is the greater root, y_i is the lesser root and k_i is the interpolator.

Step 1. First, we will start with, $x_0^2 - Dy_0^2 = k_0$, with $x_0 = 1, y_0 = 0, k_0 = 1$, and set $p_0 = 0$.

Step 2. For given values x_i, y_i, k_i and p_i, the values y_{i+1} and p_{i+1} are obtained by solving the *kuṭṭaka* (pulverizer), $y_{i+1} = \frac{y_i p_{i+1} + x_i}{|k_i|}$ with the condition $|p_{i+1}^2 - D|$ is chosen to minimum such that $y_i p_{i+1} + x_i \equiv 0 \ (mod\ |k_i|)$.

Step 3. Then x_{i+1} and k_{i+1} can be found from:

$$x_{i+1} = \frac{x_i p_{i+1} + Dy_i}{|k_i|} \text{ and}$$

$$k_{i+1} = \frac{p_{i+1}^2 - D}{|k_i|}$$

Step 4. The new equation obtained is $x_{i+1}^2 - Dy_{i+1}^2 = k_{i+1}$.

This process is repeated until one of the values for interpolator $\pm 1, \pm 2$, or ± 4 is obtained. Then one can apply the process of Samāsabhāvana or continue with Cakravāla method to get an interpolator equal to 1. If we determine the interpolator to be 1, we have solved the problem (Selenius, 1817)

Example 5. Solve $x^2 - 58y^2 = 1$, D = 58

Consider the equation $x_i^2 - 58y_i^2 = k_i$, we start with $x_0^2 - 58y_0^2 = k_0$, and $x_0 = 1, y_0 = 0, k_0 = 1$, $p_0 = 0$ & $p_1 = 8$, since $|p_1^2 - 58|$ is minimum for $p_1 = 8$

Step1. $y_1 = \frac{y_0 p_1 + x_0}{|k_0|} = \frac{0 \times p_1 + 1}{|1|} = 1$

Then $x_1 = \frac{x_0 p_1 + 58 y_0}{|k_0|} = \frac{1 \times 8 + 58 \times 0}{|1|} = 8$ and $k_1 = \frac{p_1^2 - 58}{|k_0|} = \frac{64 - 58}{1} = 6$.

Step 2. First, we solve the congruence $y_1 p_2 + x_1 \equiv 0 \ (mod |k_1|)$ for computing the value of p_2. We have

$y_1 p_2 + x_1 \equiv 0 \ (mod\ 6) \Rightarrow p_2 + 8 \equiv 0 \ (mod\ 6)$

$\Rightarrow p_2 + 2 \equiv 0 \ (mod\ 6)$

$\Rightarrow p_2 \equiv -2 \ (mod\ 6)$

$\Rightarrow p_2 \equiv 4 \ (mod\ 6)$

$\Rightarrow p_2 = 4, 16, 28, \ldots$

Here, we select that value of p_2 for which $|p_2^2 - D|$ is minimum. Hence, $|p_2^2 - D|$ is minimum for $p_2 = 4$.

$$y_2 = \frac{y_1 p_2 + x_1}{|k_1|} = \frac{1 \times 4 + 8}{|6|} = 2$$

$$x_2 = \frac{x_1 p_2 + D y_1}{|k_1|} = \frac{8 \times 4 + 58 \times 1}{|6|} = 15$$

$$k_2 = \frac{p_2^2 - D}{k_1} = \frac{4^2 - 58}{6} = -7.$$

Step 3. We solve the congruence $y_2 p_3 + x_2 \equiv 0 \ (mod |k_2|)$ for computing the value of p_3, we have

$y_2 p_3 + x_2 \equiv 0 \ (mod\ 7) \Rightarrow 2p_3 + 15 \equiv 0 \ (mod\ 7)$

$\Rightarrow 2p_3 + 1 \equiv 0 \ (mod\ 7)$

$\Rightarrow 2p_3 \equiv -1 \ (mod\ 7)$

$\Rightarrow 2p_3 \equiv 6 \ (mod\ 7) \Rightarrow p_3 \equiv 3 \ (mod\ 7)$

$\Rightarrow p_3 = 3, 10, 17, \ldots$

Here, we select that value of p_3 for which $|p_2^2 - D|$ is minimum. Hence, $|p_3^2 - D|$ is minimum for $p_3 = 10$.

$$y_3 = \frac{y_2 p_3 + x_2}{|k_2|} = \frac{2 \times 10 + 15}{|-7|} = 5$$

$$x_3 = \frac{x_2 p_3 + D y_2}{|k_1|} = \frac{15 \times 10 + 58 \times 2}{|-7|} = 38$$

$$k_3 = \frac{p_3^2 - D}{k_2} = \frac{10^2 - 58}{-7} = -6$$

Step 4. We solve the congruence $y_3 p_4 + x_3 \equiv 0 \ (mod|k_3|)$ for computing the value of p_4, we have

$y_3 p_4 + x_3 \equiv 0 \ (mod\ 6) \ \Rightarrow 5p_4 + 38 \equiv 0 \ (mod\ 6)$

$\Rightarrow 5p_4 + 2 \equiv 0 \ (mod\ 6) \Rightarrow 5p_4 \equiv -2 \ (mod\ 6)$
$\Rightarrow 5p_4 \equiv 4 (mod\ 6) \Rightarrow 25p_4 \equiv 20 \ (mod\ 6)$
$\Rightarrow p_4 \equiv 2 \ (mod\ 6) \Rightarrow p_4 = 2, 8, 14, \ldots$

Here, we select that value of p_4 for which $|p_4^2 - D|$ is minimum. Hence, $|p_4^2 - D|$ is minimum for $p_4 = 8$.

$$y_4 = \frac{y_3 p_4 + x_3}{|k_3|} = \frac{5 \times 8 + 38}{|-6|} = 13$$

$$x_4 = \frac{x_3 p_4 + D y_3}{|k_3|} = \frac{38 \times 8 + 58 \times 5}{|-6|} = 99$$

$$k_4 = \frac{p_4^2 - D}{k_3} = \frac{8^2 - 58}{-6} = 1$$

Finally, we have obtained interpolator 1. Hence, solution is given by $(x_4, y_4) = (13, 99)$. Also, we can find infinitely many integer solutions by using Brahmagupta's lemmas.

Example 6. The equation $x^2 - Dy^2 = 1$ with $D = 103$

Consider the equation $x_i^2 - 103y_i^2 = k_i$, we start with $x_0^2 - 103y_0^2 = k_0$, and $x_0 = 1, y_0 = 0, k_0 = 1, \ \ p_0 = 0$ & $p_1 = 10$, because $|p_1^2 - 103|$ is minimum for $p_1 = 10$.

Step 1. $y_1 = \frac{y_0 p_1 + x_0}{|k_0|} = \frac{0 \times p_1 + 1}{|1|} = 1$

Then,

$x_1 = \frac{x_0 p_1 + 103 y_0}{|k_0|} = \frac{1 \times 10 + 58 \times 0}{|1|} = 10$ and

$k_1 = \frac{p_1^2 - 103}{|k_0|} = \frac{100 - 103}{1} = -3$.

So, $x_1 = 10, y_1 = 1, k_1 = -3$, and $p_1 = 10$.

Step 2. First, we solve the congruence $y_1 p_2 + x_1 \equiv 0 \ (mod|-3|)$ for computing value of p_2. We have

$y_1 p_2 + x_1 \equiv 0 \ (mod\ 3) \Rightarrow p_2 + 10 \equiv 0 \ (mod\ 3)$
$\Rightarrow p_2 + 1 \equiv 0 \ (mod\ 3)$
$\Rightarrow p_2 \equiv -1 \ (mod\ 3) \Rightarrow p_2 \equiv 2 \ (mod\ 3)$
$\Rightarrow p_2 = 2, 8, 14, \ldots$

Here, we select that value of p_2 for which $|p_2^2 - D|$ is minimum. Hence $|p_2^2 - D|$ is minimum for $p_2 = 8$.

$$y_2 = \frac{y_1 p_2 + x_1}{|k_1|} = \frac{1 \times 8 + 10}{|-3|} = 6$$

$$x_2 = \frac{x_1 p_2 + D y_1}{|k_1|} = \frac{10 \times 8 + 103 \times 1}{|-3|} = 61, \ k_2 = \frac{p_2^2 - D}{k_1} = \frac{8^2 - 103}{-3} = 13.$$

And so on. We have reached at interpolator $k_{10} = 1$ after the 10th step in table 4. Hence, the smallest solution is given by $(x, y) = (227528, 22419)$.

Numerical Results for D = 103				
i	p_i	k_i	x_i	y_i
0	0	1	1	0
1	10	-3	10	1
2	8	13	61	6
3	5	-6	71	7
4	7	9	203	20
5	2	-11	274	27
6	9	2	477	47
7	11	9	5044	497
8	7	-6	9611	947
9	11	-3	33877	3338
10	10	1	227528	22419

Table 4.

5. Applications

The equation $x^2 - Dy^2 = 1$ has many applications.

- **(i)** The integer solution of the equation $x^2 - Dy^2 = 1$ provideaccurate rational approximation to $\sqrt{D}$.
- **(ii)** In the realm of integer solutions of the equation $x^2 - Dy^2 = 1$ is an active area of research in number theory and computer science (Gupta and Handa, 2013).

(iii) In the realm of modern algebra and number theory, the integer solutions $x^2 - Dy^2 = 1$ provide the domain of integer of the quadratic field $Z(\sqrt{D})$ (Dutta, 2017).

6. Conclusion

The study of indeterminate equations, especially the study of the equation $x^2 - Dy^2 = 1$ played a vital role in algebra and number theory in ancient India as well as in modern Europe. After analysing all the results, we conclude that Brahmagupta's *Bhāvanā* method provided fundamental concept in the solution of second order indeterminate equations. Jayadeva developed the cakravāla algorithm, and Bhāskara II gave a finishing touch to a wonderful algorithm called cakravāla to find the integer solutions of the second order indeterminate equations. We conclude that Bhaskara II expanded the method of solving Brahmagupta's second-order indeterminate equation. Both Bhaskara's and Brahmagupta's methods assist each other in solving second-order indeterminate equations. If we don't use Brahmagupta's auxiliary equation in the Bhaskara's cakravāla method after the satisfied step, the Bhaskara's cakravāla method becomes quite lengthy. In the process of Jayadeva and Bhāskara II, we have concluded that values of p_{i+1} are also obtained with the condition $|p_{i+1}^2 - D|$ is chosen to minimum such that $y_i p_{i+1} + x_i \equiv 0 \ (mod\ |k_i|)$. In conclusion, this research discuss the ancient techniques of solving second-order indeterminate equations, focusing on the contributions of Indian mathematicians such as Brahmagupta and Bhaskara. Through a meticulous examination of the samāsabhāvana method and the cakravāla algorithm, we have unearthed profound mathematical insights that continue to be relevant in modern algebra and number theory. By presenting examples and tabulating the results obtained through these methods, we have provided a clear and accessible explanation with their application through our investigation.

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