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Labeling of Graphs Using Taxi-Cab Metric

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Abstract: The labeling of vertices of a graph with ***m*** edges using some subset of integers from ***0*** to ***m*** inclusive, in a way that every vertex label is unique and every edge is labeled using absolute difference between its endpoints and this difference lies between ***1*** and ***m*** inclusive, is called graceful labeling. When the same technique is applied but using ordered pairs instead of numbers as identification of vertices and identification of edges are obtained using Taxi-cab metric then it is called Taxi-cab labeling of graphs. In all kinds of former labelings defined by researchers throughout the world, only single numbers are used to label the graphs but by using ordered pairs the problem to label more classes of graphs can be tackled because ordered pairs give wider range of labels per vertex to choose from. The results of this investigation verify that bipartite graphs, tripartite graphs, vertex switching of cycles and $P_{2n+1} * nP_2$ admit Taxi-cab labeling.

Keywords: Graph labeling • Taxi-cab labeling • Bipartite graph • Tripartite graph • Vertex Switching
Subject Classification (AMS): 05C78

Introduction

Graph labeling is a significant concept in graph theory, involving the allocation of identification, typically integers, to the points or lines of a graph according to specific rules or conditions. This concept has evolved since its introduction in the mid-1960s (Rosa 1967) and has numerous applications across various fields. Graph labeling can be formally defined as follows: graph $G = (V, E)$, an identification of points is a function that assigns labels from a set to the vertices V, while an edge labeling assigns labels to the edges E. The labels can be integers or other meaningful identifiers, depending on the application. There are several types of graph labelings, including: Graceful Labeling (Golomb 1972) which is an allocation of identification to the vertices from 0 to the number of lines and identification of lines are absolute differences between the identifications of the incident points. Harmonious Labeling (Graham and Sloane 1980) which is an assignment that ensures that each vertex receives a unique label, and the edge labels are derived from the vertex labels. Edge-Graceful Labeling (Sheng-Ping 1985) which is a labeling where edges are assigned distinct integers, and the vertex labels are derived from the sums of identifications of the incident edges modulo the number of vertices. Other major milestones in the said area are given in (Barrientos and Minion 2021; Gnanajothi 1991; Jeyanthi and Devi 2021; Kathiresan and Amutha 2006; Kumar et al. 2021; Kumar et al. 2024; Sun et al. 2018; Tout et al. 1982). To explore all the discovries in the aforementioned field, the exquisite survey by Gallian is reffered (Gallian 2023).

These labelings help address various combinatorial problems and have led to the study of over 200 different labeling techniques documented in extensive literature. Graph labeling serves a broad spectrum of purposes in various domains like Network Design (Prasanna 2014), in communication and computer networks, graph labeling can optimize routing and resource allocation by assigning labels that represent bandwidth, latency or other attributes. In Scheduling Problems (Ahmed 2012), graph labeling techniques can be utilized to solve scheduling issues, where tasks are represented as vertices and edges indicate dependencies or conflicts. In Coding Theory (Ahmed 2012), graph labeling helps in constructing error-correcting codes by representing code-words as labeled vertices. In Game Theory (Kearns 2013), certain games can be modeled

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using graph labeling, where strategies are represented as vertices, and the edges represent possible moves or transitions. In bio-informatics, graph labeling can be applied to model interactions in biological networks, such as protein-protein interactions or metabolic pathways. Graph labeling continues to be a rich area of research, with ongoing studies exploring new labeling techniques and their applications across different fields.

In contribution to the same cause we introduce a new type of labeling technique that uses ordered pairs instead of integers as vertex labels and the distance obtained by Taxi-cab metric between two such ordered pairs is the edge label. When the vertex labels of the form $(r,s) \in M \times M$ where, $M = \{0,1,2,\dots,m\}$ for a graph having m edges, are all distinct and the obtained edge labels using Taxi-cab metric are also distinct and belongs to the set $\{1,2,3,\dots,m\}$ the labeling is called Taxi-cab labeling of graphs. In contrast with the traditional labeling techniques, this method gives us more options per vertex as labels while also being in the range from 0 to the number of edges.

To enhance the understanding of the results presented in this paper, the following relevant definitions are provided.

Definition 1.1. A Taxi-cab metric (Shirali and Vasudeva 2005) on an Euclidean plane is defined as $d\big((c_1,d_1),(c_2,d_2)\big) = |c_1 - c_2| + |d_1 - d_2|$ for any two points $A(c_1,d_1)$ and $B(c_2,d_2)$ on the plane. It is also called Manhattan distance.

Definition 1.2. A graph is classified as bipartite, if its set of vertices can be separated into two subsets A and B that are non-empty, in order that each edge in E has a vertex in A and the other vertex in B. When the maximum possible edges are formed in this type of graph it is called complete bipartite graph (Philo et al. 2022).

Analogous to complete bipartite graphs, the complete tripartite graphs can also be defined.

Definition 1.3. Vertex switching (Ellingham 1991) in a graph refers to an operation that involves altering the connections of a specific vertex with its neighboring vertices. When vertex switching is applied to a vertex v in a graph, the incident lines to v are removed, and new lines are created to connect v to all vertices that were previously non-adjacent to it.

In the following section, we will formally define the newly introduced labeling technique. Subsequently, we will discuss the applicability of this labeling to several classes of graphs.

Results

Definition 2.1. When an injection f is defined on G of size m in a way that $f: V \to M \times M$, whilst M represents the set $\{0,1,2,\dots,m\}$ and $f^*: E \to \{1,2,3,\dots,m\}$ being a bijection defined by $f^*(uv) = |c_1 - c_2| + |d_1 - d_2|$ where, $f(u) = (c_1,d_1)$, $f(v) = (c_2,d_2)$ is induced, then this kind of labeling of graphs is defined as Taxi-cab Labeling.

According to the Definition 2.1, the subsequent findings are obtained.

Theorem 2.2. The complete bipartite graph $K_{p,q}$ can be labeled using Taxi-cab Labeling.

Proof. The graph $K_{p,q}$ has number of edges $m = pq$. The vertex set of this graph can be separated into two subsets $\{v_1, v_2, v_3, v_4, \dots, v_p\}$ and $\{u_1, u_2, u_3, u_4, \dots, u_q\}$. Define the labeling, $f: V \to M \times M$ by,

$$f(u_i) = (ip - i + 1, i - 1) \quad , 1 \le i \le q$$
$$f(v_i) = (i - 1, 0) \quad , 1 \le i \le p$$

The labeling of edges obtained $f^*: E \to \{1,2,3,\dots,m\}$ is defined by $f^*(uv) = |c_1 - c_2| + |d_1 - d_2|$ where, $(c_1,d_1),(c_2,d_2)$ are the labels of the vertices u, v respectively. We can easily see that

$f(v_1) = (0,0), f(v_2) = (1,0), \dots, f(v_p) = (p - 1, 0)$ and

$f(u_1) = (p,0), f(u_2) = (2p - 1, 1), \dots, f(u_q) = (q(p-1) + 1, q - 1)$. Here, f^* is a bijective map. Both f and f^* satisfy the condition for Taxi-cab labeling. The complete bipartite graph $K_{p,q}$ admits Taxi-cab Labeling (See Fig. 1).

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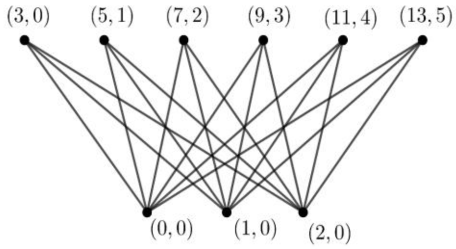

Fig. 1. Taxi-cab labeling for $K_{6,3}$.

Theorem 2.3. The complete Tripartite graph $K_{1,p,q}$ can be labeled using Taxi-cab Labeling.

Proof. The graph $K_{1,p,q}$ has number of edges $m = pq + p + q$. The vertices of this graph can be separated into three sets as: $\{a\}, \{v_1, v_2, v_3, \ldots, v_p\}$ and $\{u_1, u_2, u_3, \ldots, u_q\}$. Define the labeling, $f: V \to M \times M$ by,

$f(a) = (0,0)$

$f(u_i) = (pi + p, i) \quad , 1 \leq i \leq q$

$f(v_i) = (i, 0) \quad , 1 \leq i \leq p$

The labeling of edges obtained $f^*: E \to \{1,2,3, \ldots, m\}$ is defined by $f^*(uv) = |c_1 - c_2| + |d_1 - d_2|$ where, $(c_1, d_1), (c_2, d_2)$ are the labels of the vertices u, v respectively. Here, f^* is a bijective map. Both f and f^* satisfy the condition for Taxi-cab labeling. The complete bipartite graph $K_{1,p,q}$ admits Taxi-cab Labeling (See Fig. 2).

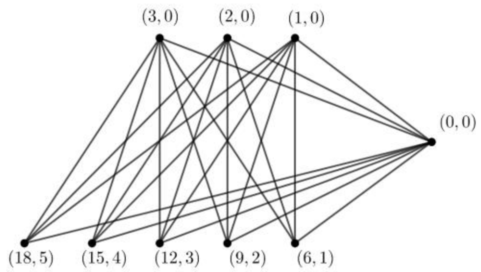

Fig. 2. Taxi-cab labeling for $K_{1,3,5}$.

Theorem 2.4. After switching an arbitrary vertex in a cycle C_n for $n > 3$, the obtained graph can be labeled using Taxi-cab Labeling.

Proof. Assume that the set of vertices of C_n be $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and *WLOG* assume that the vertex v_0 is switched. The obtained graph has number of edges $m = 2n - 5$. Define the vertex labeling function, $f: V \to M \times M$ by,

Condition 1: When $n =$ even,

$$f(v_0) = (0,0)$$
$$f(v_1) = (n-3,1)$$
$$f(v_i) = \left(2n - \frac{3i}{2} - 3, \frac{i}{2}\right) \quad , i \text{ is even}$$
$$f(v_i) = \left(\frac{i-1}{2}, \frac{i-3}{2}\right) \quad , i \text{ is odd}$$

Condition 2: When $n =$ odd,

$$f(v_0) = (0,0)$$
$$f(v_{n-1}) = (n-3,0)$$
$$f(v_i) = \left(\frac{n-i-1}{2}, \frac{n-i-3}{2}\right) \quad , i \text{ is even}$$
$$f(v_i) = \left(\frac{n+3i}{2} - 2, \frac{n-i}{2} - 1\right) \quad , i \text{ is odd}$$

The labeling of edges obtained $f^*: E \to \{1,2,3,\ldots,m\}$ is defined by $f^*(uv) = |c_1 - c_2| + |d_1 - d_2|$ where, $(c_1, d_1), (c_2, d_2)$ are the labels of the vertices u, v respectively. Here, f^* is a bijective map. Both f and f^* satisfy the condition for Taxi-cab labeling. So, the vertex switched graph of C_n admits Taxi-cab Labeling (See Fig. 3).

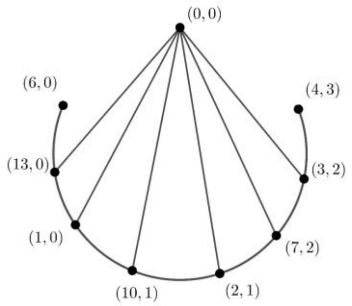

Fig. 3. Taxi-cab labeling for vertex switched C_9 graph.

Theorem 2.5. The graph $P_{2n+1} * nP_2$ can be labeled using Taxi-cab Labeling.

Proof. Let the vertices of P_{2n+1} be $\{v_1, v_2, \ldots, v_{2n+1}\}$, represent the vertices $v_{n+2}, v_{n+3}, v_{n+4}, \ldots, v_{2n+1}$ by $u_n, u_{n-1}, u_{n-2}, \ldots, u_1$ respectively, the graph $P_{2n+1} * nP_2$ is formed by joining the vertices v_i by u_i for $1 \le i \le n$, v_n by v_{n+1} and v_{n+1} by u_n. The obtained graph has number of edges $m = 3n$. Define the vertex labeling function, $h: V \to M \times M$ by,

$$h(v_i) = \left(m + 1 - i, \frac{i-1}{2}\right) \quad , i = 1,3,5,\ldots,n+1$$

$$h(v_i) = \left(1, \frac{i-2}{2}\right) \quad , i = 2,4,6,\ldots,n+1$$

$$h(u_i) = \left(m - n, \frac{i-1}{2}\right) \quad , i = 3,5,7,\ldots,n$$

$$h(u_i) = \left(n + 2 - i, \frac{i-2}{2}\right) \quad , i = 2,4,6,\ldots,n$$

$$h(u_1) = (0,0)$$

The labeling of edges obtained $h^*: E \to \{1,2,3,\ldots,m\}$ is defined by $h^*(uv) = |c_1 - c_2| + |d_1 - d_2|$ where, $(c_1, d_1), (c_2, d_2)$ are the labels of the vertices u, v respectively. Here, h^* is a bijective map. Both h and h^* satisfy the condition for Taxi-cab labeling. So, the graph $P_{2n+1} * nP_2$ admits Taxi-cab Labeling (See Fig. 4).

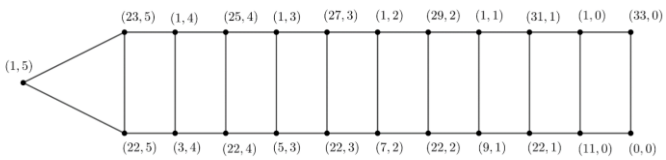

Fig. 4. Taxi-cab labeling for $P_{23} * 11P_2$.

Conclusion

As graph labeling continues to evolve, further investigations into new labeling techniques and their potential applications are essential. The Taxi-cab labeling represent a promising direction in this field. It not only contributes to theoretical advancements but also has real world implications across various sectors. Next steps in research should involve perfecting these techniques and looking into other potential applications, thereby enhancing the utility and understanding of graph labeling in both academic and practical contexts and also on developing more efficient algorithms and exploring the relationships between various labeling types to enhance our understanding and application of graph theory. The outcomes from this investigation contribute to the growing horizon of comprehension in graph labeling

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and deliver a foundation for forthcoming explorations in this dynamic area of research.

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