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The Coefficients of Bell Polynomials and Arithmetic Functions: Applicable in Fourier series and Fourier Transforms

Hemant Kumar [1] • MA Pathan [2] • J López-Bonilla [3]

[1]*Department of Mathematics, D. A-V. Postgraduate College, Kanpur 208001, Uttar Pradesh, India;*
[2]*Centre for Mathematical and Statistical Sciences, Peechi Campus, Peechi 680 653, Kerala, India;*
[3]*ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738,CDMX, México*

*Corresponding Author Email id: palhemant2007@rediffmail.com

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Abstract: In this article, we consider the polynomials consisting of the coefficients of complete and the partial Bell polynomials and the polynomial expressions for the arithmetic functions $t_k(n)$ and $r_k(n)$, the number of representations of n as a sum of k triangular numbers and k squares, respectively, and also the color partitions $p_k(n)$. Then making an appeal to the Fourier series consisting of the coefficients of Bell polynomials and arithmetic functions we obtain various results concerning approximations and Fourier transformation identities.
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1. Introduction and preliminaries

If two analytic functions $F(q)$ and $G(q)$ of q, where $|q|<1$, are followed the equations

$$q\frac{d}{dq}\log F(q) = G(q),$$
$$(F(q))^k = \sum_{n=0}^{\infty} f_k(n)\, q^n,$$
$$G(q) = \sum_{n=1}^{\infty} g_n\, q^n, \tag{1}$$

with the values $F(0)=1$, $G(0)=0$.

Then, due to Eqn. (1) there exists following recurrence relations [1]

$$f_k(n) = \frac{k}{n}\sum_{j=1}^{n} g_j\, f_k(n-j), \tag{2}$$

$$g_n = -\, n\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k} f_k(n). \tag{3}$$

The relation (3) is the inversion of (2) with respect to the sequence $\{g_n\}$.

From the hypotheses $f_k(0)=1,\ k\geq 0,\ f_0(n)=\delta_{0n},\ g_0=0$, $f_k(n)$ be a polynomial in k of degree

$$f_k(n) = a(n,n)\, k^n + a(n,n-1)\, k^{n-1} + \cdots + a(n,2)\, k^2 + a(n,1)\, k, \qquad n\geq 1, \tag{4}$$

where in (4) the coefficients $a(n,m)$ are determined in terms of the quantities g_j, given by

$$a(n,n) = \frac{1}{n!}(g_1)^n, \quad n\geq 1, \tag{5}$$

$$a(n,m) = \frac{1}{m!\,(n-m)!} \sum_{j=1}^{n-m} (g_1)^{m-j} \binom{m}{j} j!\; B_{n-m,\,j}\left(\frac{1!}{2}\, g_2\,,\, \frac{2!}{3}\, g_3\,, \ldots, \frac{(n-m-j+1)!}{n-m-j+2}\, g_{n-m-j+2}\right),$$
$$n \geq m+1, \tag{6}$$

involving the partial exponential Bell polynomials [5, 6, 21]. Here the results (5) and (6) are in harmony with the relations due to Jakimczuk [10, Eqns. (8)-(11)].

2. Fourier series involving the coefficients of complete and partial Bell polynomials

Lemma 2.1 *In* [23], *due to* (4) *the coefficients of the partial Bell polynomial are represented by the Eqns*. (5) and (6).

Proof. Consider the polynomial (4) which has a continuous variable in k, then on applying $\frac{d^m}{dk^m}$ to the property
$F^k = \sum_{n=0}^{\infty} f_k(n)\, q^n$ and after to make $k=0$, there exists following equalities [23]

$$\sum_{n=0}^{\infty} m!\; a(n,m) q^n = (\log F)^m = \left(\sum_{n=1}^{\infty} \frac{g_n}{n} q^n\right)^m = (g_1\, q)^m \left(\sum_{n=0}^{\infty} h_n \frac{q^n}{n!}\right)^m, \; h_0 = 1,$$
$$h_n = \frac{n!\, g_{n+1}}{(n+1) g_1} = \quad (g_1\, q)^m \sum_{j=0}^{\infty} Q_j\, \frac{q^j}{j!} = (g_1)^m \sum_{n=m}^{\infty} Q_{n-m}\, \frac{q^n}{(n-m)!}\,, \tag{7}$$

which due to [17, 23] give

$$Q_0 = 1, \qquad Q_{n-m} = \sum_{l=1}^{n-m} \binom{m}{l}\, l!\; B_{n-m,l}(h_1\,, h_2\,, \ldots, h_{n-l+1}), \qquad n - m \geq 1. \tag{8}$$

Hence by the Eqns. (7) and (8), the coefficients are

$$a(n,m) = \begin{cases} 0, & 0 \leq n \leq m-1, \\ \frac{(g_1)^m}{m!}, & n = m, \\ \frac{(g_1)^m}{m!\,(n-m)!}\, Q_{n-m}\,, & m+1 \leq n. \end{cases} \tag{9}$$

Therefore (7) - (9) imply (5) and (6).

Theorem 2.2 *For all* $m, r \in \mathbb{Z}_+$, $\mathbb{Z}_+ =$ (*a set of positive integers*) *and* $x \in (-\infty, \infty)$ *define a Fourier series such that*

$$H\,(m,r;\; x) = \sum_{n=-\infty}^{\infty} C(n,m,r)\, e^{inx}, i = \sqrt{(-1)}, \tag{10}$$

then for any $r > 0$, *and due to* (10) *there exists a formula*

$$C(r,m,r) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty} H\,(m,r;\; x) e^{-irx} dx\,; n = r, r > 0, m > 0 \\ 0; n \neq r, r > 0, m > 0 \end{cases} \tag{11}$$

Proof. Consider the series (10) and multiply in its both sides by e^{-irx}, for any fixed $r > 0, x \in (-\infty, \infty)$, then integrate that sides with respect to x from $-\infty$ to ∞ to get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H\,(m,r;\; x) e^{-irx} dx$$
$$= \sum_{n=-\infty}^{\infty} C(n,m,r) \left[\frac{1}{2\pi}\int_{-\infty}^{-\pi} e^{i(n-r)x}\, dx + \frac{1}{2\pi}\int_{\pi}^{\infty} e^{i(n-r)x}\, dx\right]$$

$$+\sum_{n=-\infty}^{\infty} C(n,m,r)\,\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-r)x}\,dx \tag{12}$$

By the Eqn. (12) and due to the formulae used in [3, 4, 13], we immediately find the result (11).

Theorem 2.3 *In the series* (10) *of the Theorem* 2.2, *if suppose that*

$$C(n,m,n) = \begin{cases} a(n,m), n \geq m+1 \\ a(m,m), n = m \\ 0, m-1 \geq n \geq 0 \\ 0, n < 0 \end{cases} \tag{13}$$

Then $\forall x \in [0,2\pi]$, *there exists an inequality*

$$\left|2m!\,H\,(m;\ x) - \sum_{n=m+1}^{\infty} \frac{2m!(g_1)^m}{m!\,(n-m)!}\,Q_{n-m}\,e^{inx}\right| \leq \sqrt{\left\{\binom{2m}{m}\right\}}(g_1)^m \tag{14}$$

Proof. Consider a series under the conditions given in (13) and make an appeal to equalities (9), we write it as

$$H\,(m;\ x) = \sum_{n=-\infty}^{\infty} C(n,m,n)\,e^{inx}$$
$$= \sum_{n=-\infty}^{-1} 0\,e^{inx} + \sum_{n=0}^{m-1} 0\,e^{inx} + \sum_{n=m} a(n,m)\,e^{inx} + \sum_{n=m+1}^{\infty} a(n,m)\,e^{inx}$$
$$= a(m,m)e^{imx} + \sum_{n=m+1}^{\infty} a(n,m)\,e^{inx} \tag{15}$$

Now in series (15) use the equalities (9) that implies as

$$\left|H\,(m;\ x) - \sum_{n=m+1}^{\infty} \frac{(g_1)^m}{m!\,(n-m)!}\,Q_{n-m}\,e^{inx}\right|^2 \leq \frac{(g_1)^{2m}}{m!m!}\left|e^{2mix}\right|. \tag{16}$$

Finally, $\forall x \in [0,2\pi]$ and by the inequality (16), we arrive on the result (14).

Theorem 2.4 *If all conditions of the Theorem* 2.3 *are followed, then for all* $x \in (-\infty,\infty)$ *and* $m \in \mathbb{Z}_+$ *there exists a formula*

$$\int_{-\infty}^{\infty} H\,(m;\ x)\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}\,dx = \sum_{n=m}^{\infty} \frac{(g_1)^m}{m!\,(n-m)!}\,Q_{n-m}\sum_{j=1}^{q} \frac{\pi c\exp[-b_j n]}{b_j\prod_{l=1,l\neq j}^{q}(-b_j^2+b_l^2)}. \tag{17}$$

Proof. For all $x \in (-\infty,\infty)$ consider a function

$$f(x) = \frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}$$

and another function $H\,(m;\ x), \forall m \in \mathbb{Z}_+$ and $x \in (-\infty,\infty)$ from the Eqn. (15), we get

$$\int_{-\infty}^{\infty} H\,(m;\ x)\,f(x)\,dx = \int_{-\infty}^{\infty} a(m,m)\,e^{imx}\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}dx + \sum_{n=m+1}^{\infty}\int_{-\infty}^{\infty} a(n,m)\,e^{inx}\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}\,dx$$
$$\Rightarrow \int_{-\infty}^{\infty} H\,(m;\ x)\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}\,dx$$
$$= \frac{(g_1)^m}{m!}\int_{-\infty}^{\infty} e^{imx}\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}dx + \sum_{n=m+1}^{\infty} \frac{(g_1)^m}{m!\,(n-m)!}\,Q_{n-m}\int_{-\infty}^{\infty} e^{inx}\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}\,dx$$

Now use the *Cauchy Residue Theorem* [22] and the techniques of [12, 28] we evaluate

$$\int_{-\infty}^{\infty} H\,(m;\ x)\,\frac{c}{\prod_{j=1}^{q}(x^2+b_j^2)}\,dx$$
$$= \pi c\,\frac{(g_1)^m}{m!}\sum_{j=1}^{q} \frac{\exp[-b_j m]}{b_j\prod_{l=1,l\neq j}^{q}(-b_j^2+b_l^2)} + \pi c\sum_{n=m+1}^{\infty} \frac{(g_1)^m}{m!\,(n-m)!}\,Q_{n-m}\sum_{j=1}^{q} \frac{\exp[-b_j n]}{b_j\prod_{l=1,l\neq j}^{q}(-b_j^2+b_l^2)} \tag{18}$$

The Eqn. (18) immediately gives us the formula (17).

It is remarked that on making an appeal to the techniques and results of [12, 28], the formula (17) become very useful in evaluation of various trigonometrically functions and polynomials.

3. Arithmetic functions their partial and complete Bell polynomials and Fourier transformation identities

Various authors to them for example [16-18] studied the arithmetic functions $r_k(n)$ and $t_k(n)$, which are the number of representations of n as a sum of k squares and k triangular numbers, respectively. The function $p_k(n)$ is the number of color partitions of n.
Again, we recall the function due to [1, 23]

$$F(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} \frac{1-q^n}{1+q^n} \tag{19}$$

Here in (19) consider that

$$F^k = \sum_{n=0}^{\infty} (-1)^n r_k(n)\, q^n, \qquad G(q) =-2 \sum_{n=1}^{\infty} n\, D(n)\, q^n. \tag{20}$$

In the relations (20), $D(n)$ is the sum of the inverses of the odd divisors of n, that is, $D(n) = \sum_{odd\ d/n} \frac{1}{d}$, and as $r_k(n)$ is the number of representations of a positive integer n as a sum of k squares, such that representations with different orders and signs are counted as distinct [8, 9, 14, 19]. Therefore following relations are found [23]

$$\begin{aligned} f_k(n) &= (-1)^n r_k(n), \\ g_n &=-2n\, D(n) = n\, g_1\, D(n), \quad g_1 =-2, \end{aligned} \tag{21}$$

whose application into (4)-(6) gives the expression

$$r_k(n) = \frac{2^n}{n!} k^n + (-1)^n \sum_{j=1}^{n-1} a(n,j)\, k^j ,$$

such that

$$\begin{aligned} a(n,m) = \frac{(-2)^m}{m!\,(n-m)!} \sum_{j=1}^{n-m} \binom{m}{j} j! \\ \times B_{n-m,j}(1!\,D(2), 2!\,D(3), \dots, (n-m-j+1)!\,D(n-m-j+2)). \end{aligned} \tag{22}$$

Again consider that [11, 23]

$$F(q) = (q\,;q)_\infty = \prod_{n=1}^{\infty} (1-q^n) , \tag{23}$$

$$F^k = \sum_{n=0}^{\infty} p_k(n)\, q^n , \; G(q) =-\sum_{n=1}^{\infty} \sigma(n)\, q^n , \tag{24}$$

where $p_k(n)$ is the number of partitions [10, 15] with k colors (see [16-18]) and $\sigma(n)$ is the sum of the divisors of n; hence there are the relations (also see in [23])

$$f_k(n) = p_k(n), \qquad g_n =-\sigma(n) = g_1\, \sigma(n), \quad g_1 =-1, \tag{25}$$

and (4)-(6) imply the relation

$$p_k(n) = \frac{(-1)^n}{n!} k^n + \sum_{j=1}^{n-1} a(n,j)\, k^j , \tag{26}$$

such that

$$a(n,m) = \frac{(-1)^m}{m!\,(n-m)!} \sum_{j=1}^{n-m} \binom{m}{j} j!\; B_{n-m,j}\left(\frac{1!}{2}\,\sigma(2), \frac{2!}{3}\,\sigma(3), \ldots, \frac{(n-m-j+1)!}{n-m-j+2}\,\sigma(n-m-j+2)\right), \tag{27}$$

further consider that

$$F(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^{2n-1}} = \prod_{n=1}^{\infty} (1+q^n)^2 (1-q^n), \tag{28}$$

and

$$F^k = \sum_{n=0}^{\infty} t_k(n)\, q^n, \quad G(q) = -\sum_{n=1}^{\infty} n\, T(n)\, q^n, \tag{29}$$

where $t_k(n)$ is the number of representations of n as the sum of k triangular numbers, such that representations with different orders are counted as unique, and:

$$T(j) = \sum_{d|j} \frac{1+2(-1)^d}{d} = \frac{1}{j} \sum_{d|j} (-1)^d\, d, \tag{30}$$

hence

$$f_k(n) = t_k(n). \qquad g_n = -n\,T(n), \quad g_1 = 1, \tag{31}$$

and from (4)-(6)

$$t_k(n) = \frac{1}{n!} k^n + \sum_{j=1}^{n-1} a(n,j)\, k^j, \tag{32}$$

where

$$a(n,m) = \frac{1}{m!\,(n-m)!} \sum_{j=1}^{n-m} (-1)^j \binom{m}{j} j!$$
$$\times B_{n-m,j}\big(1!\,T(2), 2!\,T(3), \ldots, (n-m-j+1)!\,T(n-m-j+2)\big). \tag{33}$$

Due to the Eqns. (21)-(33), here in our research work, we use following interesting recurrence relations [23] to obtain Fourier series identities

$$p_{k+1}(n) = \sum_{j=0}^{n} a(j)\, p_k(n-j), \tag{34}$$

provided that

$$a(j) = \begin{cases} 0, & j \neq \frac{m}{2}(3m+1), m = 0, \pm 1, \pm 2, \ldots \\ (-1)^m, & j = \frac{m}{2}(3m+1), m = 0, \pm 1, \pm 2, \ldots \end{cases} \tag{35}$$

$$r_{k+1}(n) = \sum_{j=0}^{n} b(j)\, r_k(n-j), \tag{36}$$

provided that

$$b(j) = \begin{cases} 2, & j = m^2,\ m \geq 1, \\ 1, & j = 0 \\ 0, & otherwise. \end{cases} \tag{37}$$

$$t_{k+1}(n) = \sum_{j=0}^{n} c(j)\, t_k(n-j), \tag{38}$$

provided that

$$c(j) = \begin{cases} 1, & j = \frac{m}{2}(m+1), \quad m \geq 0, \\ 0, & otherwise. \end{cases} \tag{39}$$

Now we present some identities on Fourier transformations due to arithmetic functions defined in the Eqns. (34) to (39).

Theorem 3.1 *If the*

$$H_1(x;k,L) = \sum_{n=-\infty}^{\infty} C_{n,k}^{(1)} exp\left[\frac{2n\pi ix}{L}\right], \tag{40}$$

where, $C_{n,k}^{(1)} = \begin{cases} p_{k+1}(n), n \geq 0; \\ 0, n < 0. \end{cases}$

$i = \sqrt{(-1)}, L > 0, \forall x \in \mathbb{R}$ *{a set of real numbers},* $k \in \mathbb{Z}_+$ *{a set of positive integers},* (41)

then $\forall N \geq \frac{m}{2}(3m+1)$, *where* $m = 0, \pm 1, \pm 2, \ldots$, *there exists following identical Fourier transforms*

$$\frac{1}{L}\int_{-L/2}^{L/2} H_1(x;k,L) e^{-\frac{2\pi Nix}{L}} dx = \frac{e^{im\pi}}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi\left(N-\frac{m}{2}(3m+1)\right)ix}{L}\right] H_1(x;k-1,L)\,dx. \tag{42}$$

Proof. Making an appeal to the recurrence relation of the arithmetic function $p_k(n)$ given in (34) and the definition of Fourier series in (40)-(41), we find

$$H_1(x;k,L) = \sum_{n=0}^{\infty} p_{k+1}(n) e^{\frac{2n\pi ix}{L}} = \sum_{n=0}^{\infty} e^{\frac{2n\pi ix}{L}} \sum_{j=0}^{n} a(j)\, p_k(n-j) = \sum_{n=0}^{\infty} p_k(n) e^{\frac{2n\pi ix}{L}} \sum_{j=0}^{\infty} a(j)\, e^{\frac{2j\pi ix}{L}}$$

$$\Rightarrow \frac{1}{L}\int_{-L/2}^{L/2} H_1(x;k,L) e^{-\frac{2N\pi ix}{L}} dx = \sum_{j=0}^{\infty} a(j) \frac{1}{L}\int_{-L/2}^{L/2} e^{-\frac{2(N-j)\pi ix}{L}} \left\{\sum_{n=0}^{\infty} p_k(n) e^{\frac{2n\pi ix}{L}}\right\} dx. \tag{43}$$

Then in (43) use the definition (35) to obtain

$$\frac{1}{L}\int_{-L/2}^{L/2} H_1(x;k,L) e^{-\frac{2\pi Nix}{L}} dx = (-1)^m \frac{1}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi\left(N-\frac{m}{2}(3m+1)\right)ix}{L}\right] H_1(x;k-1,L)\,dx,$$

provided that

$$\forall N \geq \frac{m}{2}(3m+1); m = 0, \pm 1, \pm 2, \ldots \quad . \tag{44}$$

The Eqn. (44) immediately gives the identity (42).

Theorem 3.2 *If the*

$$H_2(x;k,L) = \sum_{n=-\infty}^{\infty} C_{n,k}^{(2)} exp\left[\frac{2n\pi ix}{L}\right], \tag{45}$$

where, $C_{n,k}^{(2)} = \begin{cases} r_{k+1}(n), n \geq 0; \\ 0, n < 0. \end{cases}$

$i = \sqrt{(-1)}, L > 0, \forall x \in \mathbb{R}$ *{a set of real numbers},* $k \in \mathbb{Z}_+$ *{a set of positive integers},* (46)

then $\forall N \geq m^2$, *where* $m \geq 1$, *there exists following identical Fourier transforms*

$$\frac{1}{L}\int_{-L/2}^{L/2} H_2(x;k,L) e^{-\frac{2\pi Nix}{L}} dx = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi Nix}{L}\right] \left\{1 + 2\exp\left[\frac{2\pi m^2 ix}{L}\right]\right\} H_2(x;k-1,L)\,dx. \tag{47}$$

Proof. Making an appeal to the recurrence relation of the arithmetic function $r_k(n)$ given in (36) and the definition of Fourier series (45)-(46), we have

$$H_2(x;k,L) = \sum_{n=0}^{\infty} r_{k+1}(n) e^{\frac{2n\pi i x}{L}} = \sum_{n=0}^{\infty} e^{\frac{2n\pi i x}{L}} \sum_{j=0}^{n} b(j) r_k(n-j) = \sum_{n=0}^{\infty} r_k(n) e^{\frac{2n\pi i x}{L}} \sum_{j=0}^{\infty} b(j)\, e^{\frac{2j\pi i x}{L}}$$

$$\Rightarrow \frac{1}{L}\int_{-L/2}^{L/2} H_2(x;k,L) e^{-\frac{2N\pi i x}{L}} dx = \sum_{j=0}^{\infty} b(j) \frac{1}{L}\int_{-L/2}^{L/2} e^{-\frac{2(N-j)\pi i x}{L}} \left\{\sum_{n=0}^{\infty} r_k(n) e^{\frac{2n\pi i x}{L}}\right\} dx. \tag{48}$$

Then in (48) use the definition (37), we obtain

$$\frac{1}{L}\int_{-L/2}^{L/2} H_2(x;k,L) e^{-\frac{2\pi N i x}{L}} dx = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi N i x}{L}\right] H_2(x;k-1,L) dx$$
$$+ \frac{2}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi(N-m^2) i x}{L}\right] H_2(x;k-1,L) dx,$$

provided that $\forall N \geq m^2; m \in \mathbb{Z}^+, \mathbb{Z}^+ =$ {set of positive integers}. (49)

The Eqn. (49) immediately gives the identity (47).

Theorem 3.3 *If the*

$$H_3(x;k,L) = \sum_{n=-\infty}^{\infty} C_{n,k}^{(3)} exp\left[\frac{2n\pi i x}{L}\right], \tag{50}$$

where, $C_{n,k}^{(3)} = \begin{cases} t_{k+1}(n), n \geq 0; \\ 0, n < 0, \end{cases}$

$i = \sqrt{(-1)}, L > 0, \forall x \in \mathbb{R}$ {*a set of real numbers*}, $k \in \mathbb{Z}_+$ {*a set of positive integers*}, (51)

then $\forall N \geq \frac{m}{2}(m+1)$, *where* $m \in \mathbb{Z}_+ \cup \{0\}$, *there exists following identical Fourier transforms*

$$\frac{1}{L}\int_{-L/2}^{L/2} H_3(x;k,L) e^{-\frac{2\pi N i x}{L}} dx = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi\left(N-\frac{m}{2}(m+1)\right) i x}{L}\right] H_3(x;k-1,L) dx. \tag{52}$$

Proof. Making an appeal to the recurrence relation of the function $t_k(n)$ in (38) and the definition of Fourier series in (50)-(51), we get

$$H_3(x;k,L) = \sum_{n=0}^{\infty} t_{k+1}(n) e^{\frac{2n\pi i x}{L}} = \sum_{n=0}^{\infty} e^{\frac{2n\pi i x}{L}} \sum_{j=0}^{n} c(j) t_k(n-j) = \sum_{n=0}^{\infty} t_k(n) e^{\frac{2n\pi i x}{L}} \sum_{j=0}^{\infty} c(j)\, e^{\frac{2j\pi i x}{L}}$$

$$\Rightarrow \frac{1}{L}\int_{-L/2}^{L/2} H_3(x;k,L) e^{-\frac{2N\pi i x}{L}} dx = \sum_{j=0}^{\infty} c(j) \frac{1}{L}\int_{-L/2}^{L/2} e^{-\frac{2(N-j)\pi i x}{L}} \left\{\sum_{n=0}^{\infty} t_k(n) e^{\frac{2n\pi i x}{L}}\right\} dx. \tag{53}$$

Then in (53) use the definition (39), we obtain

$$\frac{1}{L}\int_{-L/2}^{L/2} H_3(x;k,L) e^{-\frac{2\pi N i x}{L}} dx = \frac{1}{L}\int_{-L/2}^{L/2} \exp\left[-\frac{2\pi\left(N-\frac{m}{2}(m+1)\right) i x}{L}\right] H_3(x;k-1,L) dx,$$

provided that

$\forall N \geq \frac{m}{2}(m+1)$, *where* $m \in \mathbb{Z}_+ \cup \{0\}$. (54)

Hence the identity (52) exists.

4 Generalization of the recurrence formula (2) and certain results

Consider the formula (2) in the form

$$n\, f_k(n) = \sum_{j=1}^{n} h(j) f_k(n-j), \tag{55}$$

and write the Eqn. (55) in the form

$$y_n = \sum_{j=1}^{n} \binom{n-1}{j-1} x_j\, y_{n-j}, \tag{56}$$

with $y_n = n!\ f_k(n)$ and $x_j = (j-1)!\ h(j)$; then (56) implies the following expression

$$y_n = B_n(x_1, \ldots, x_n), \tag{57}$$

in terms of the complete Bell polynomials [26] and therefore we have

$$f_k(n) = \frac{1}{n!} B_n\big(0!\,h(1), 1!\,h(2), \ldots, (n-1)!\,h(n)\big). \tag{58}$$

On the other hand, in [1] were proved the recurrence relations

$$r_k(n) = -\frac{2k}{n} \sum_{j=1}^{n} (-1)^j\ j\,D(j)\, r_k(n-j),$$

$$t_k(n) = -\frac{k}{n} \sum_{j=1}^{n} j\ T(j)\, t_k(n-j),$$
$$p_k(n) = -\frac{k}{n} \sum_{j=1}^{n} \sigma(j)\ p_k(n-j), \tag{59}$$

with the structure (2), which is a particular case of (59); then the application of (55) to (59) gives the interesting relations

$$r_k(n) = \frac{1}{n!} B_n\big(2k\cdot 1!\,D(1), -2k\cdot 2!\,D(2), 2k\cdot 3!\,D(3), \ldots, -2k(-1)^n\cdot n!\,D(n)\big),$$

$$t_k(n) = \frac{1}{n!} B_n\big(-k\cdot 1!\,T(1), -k\cdot 2!\,T(2), -k\cdot 3!\,T(3), \ldots, -k\cdot n!\,T(n)\big),$$
$$p_k(n) = \frac{1}{n!} B_n\big(-k\cdot 0!\,\sigma(1), -k\cdot 1!\,\sigma(2), -k\cdot 2!\,\sigma(3), \ldots, -k\cdot(n-1)!\,\sigma(n)\big), \tag{60}$$

which are closed expressions with the participation of the complete Bell polynomials. From (60) it is evident that these arithmetical functions are polynomials in k of degree n, in accordance with (4). Similarly, the known relation for the partition function [24, 25]

$$n\,p(n) = \sum_{j=1}^{n} \sigma(j)\ p(n-j), \tag{61}$$

with the structure (55)-(57), implies the property

$$p(n) = \frac{1}{n!} B_n\big(0!\ \sigma(1), 1!\ \sigma(2), 2!\ \sigma(3), \ldots, (n-1)!\ \sigma(n)\big), \tag{62}$$

in accordance with the result in [19, Theorem 7]. Besides, Robbins [24, 25] deduced the following recurrence relation

$$n\,p_D(n) = \sum_{j=1}^{n} \sigma_o(j)\ p_D(n-j), \tag{63}$$

similar to (61), where $p_D(n)$ is the number of partitions of n using only distinct parts and given as

$$\sigma_o(n) = \sum_{odd\ d|n} d = \sum_{d|n} (-1)^{d-1} \frac{n}{d} \ ; \tag{64}$$

then it is immediate the expression

$$p_D(n) = \frac{1}{n!} B_n\big(0!\ \sigma_o(1), 1!\ \sigma_o(2),\ 2!\ \sigma_o(3), \ldots, (n-1)!\ \sigma_o(n)\big). \tag{65}$$

Finally, from (63) it is possible to obtain the identity [24-25] as

$$\sigma_o(n) = \sum_{j=1}^{n} (-1)^{j-1}\, j\, p_{OD}(j)\, p_D(n-j), \tag{66}$$

where $p_{OD}(n)$ is the number of partitions of n into parts which are odd and distinct.

5 On a recurrence relation of Apostol [2, 24]

In this section, we recall the Eqns. (1) - (3) to consider the case

$$F^k = 1 + \sum_{n=1}^{\infty} f_k(n)\, q^n = \prod_{n=1}^{\infty} (1-q^n)^{-\frac{k}{n}Q(n)}, \tag{67}$$

we obtain

$$G(q) = \sum_{n=1}^{\infty} \frac{Q(n)}{1-q^n}\, q^n \overset{[11]}{=} \sum_{n=1}^{\infty} \left(\sum_{d|n} Q(d)\right) q^n, \tag{68}$$

therefore (2) implies the recurrence relation of Apostol [2, 24]

$$n\, f_k(n) = k \sum_{j=1}^{n} \left(\sum_{d|j} Q(d) \right) f_k(n-j). \tag{69}$$

If $Q(n) = -n$, then $f_k(n) = p_k(n)$ and $\sum_{d|n} Q(d) = -\ \sigma(n)$, thus (69) implies (61)

$$n\, p_k(n) = -\ k \sum_{j=1}^{n} \sigma(j)\, p_k(n-j),$$

which was originally found by Gandhi [7, 15], and for $k = -1$ it gives (63), and if $k = 1$ it generates the identity obtained by Robbins [24] and Osler-Hassen-Chandrupatla [20]

$$\sigma(n) = -n\, a(n) - \sum_{j=1}^{n-1} \sigma(j)\, a(n-j) \qquad n \geq 2, \tag{70}$$

with the $a(j)$ defined in (35).

In [23] there is also a study of polynomial expression (4) and its coefficients.

6 Concluding remarks

The Fourier series is the summation of trigonometrically functions found in the literature and applicable in various scientific problems for example see in [3, 4, 13]. We observe that the recurrence relations obtained by own techniques on applying Bell polynomials become helpful to express the Fourier transformation identities obtained in the *Section* 3.

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