The Coefficients of Bell Polynomials and Arithmetic Functions: Applicable in Fourier series and Fourier Transforms

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Abstract: In this article, we consider the polynomials consisting of the coefficients of complete and the partial Bell polynomials and the polynomial expressions for the arithmetic functions \( t_k(n) \) and \( r_k(n) \), the number of representations of \( n \) as a sum of \( k \) triangular numbers and \( k \) squares, respectively, and also the color partitions \( p_k(n) \). Then making an appeal to the Fourier series consisting of the coefficients of Bell polynomials and arithmetic functions we obtain various results concerning approximations and Fourier transformation identities.

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1. Introduction and preliminaries

If two analytic functions \( F(q) \) and \( G(q) \), where \( |q| < 1 \), are followed the equations

\[
q \frac{d}{dq} \log F(q) = G(q),
\]

\[
(F(q))^k = \sum_{n=0}^{\infty} f_k(n) q^n,
\]

\[
G(q) = \sum_{n=1}^{\infty} g_n q^n,
\]

with the values \( F(0) = 1, G(0) = 0. \)

Then, due to Eqn. (1) there exists following recurrence relations [1]

\[
f_k(n) = \frac{k}{n} \sum_{j=1}^{n} g_j f_k(n-j),
\]

\[
g_n = -n \sum_{k=1}^{n} (-1)^k \binom{n}{k} f_k(n).
\]

The relation (3) is the inversion of (2) with respect to the sequence \( \{g_n\} \).

From the hypotheses \( f_k(0) = 1, k \geq 0, f_0(n) = \delta_{0n}, g_0 = 0, f_k(n) \) be a polynomial in \( k \) of degree

\[
f_k(n) = a(n,n) k^n + a(n, n-1) k^{n-1} + \cdots + a(n, 2) k^2 + a(n, 1) k, \quad n \geq 1,
\]

(4)

where in (4) the coefficients \( a(n,m) \) are determined in terms of the quantities \( g_j \), given by

\[
a(n,n) = \frac{1}{n!} (g_1)^n, \quad n \geq 1,
\]

(5)
involving the partial exponential Bell polynomials [5, 6, 21]. Here the results (5) and (6) are in harmony with the relations due to Jakimczuk [10, Eqns. (8)-(11)].

1 Fourier series involving the coefficients of complete and partial Bell polynomials

**Lemma 1.1** In [23], due to (4) the coefficients of the partial Bell polynomial are represented by the Eqns. (5) and (6).

**Proof.** Consider the polynomial (4) which has a continuous variable in \(k\). Then on applying \(\frac{d^m}{dk^m}\) to the property

\[
F^k = \sum_{n=0}^{\infty} f_k(n) q^n
\]

and after to make \(k = 0\), there exists following equalities [23]

\[
\begin{align*}
\sum_{n=0}^{\infty} m! \ a(n, m) q^n &= (\log F)^m = \left(\sum_{n=1}^{\infty} \frac{g_n}{n} q^n\right)^m = (g_1 q)^m \left(\sum_{n=1}^{\infty} h_n \frac{q^n}{n!}\right)^m, \quad h_0 = 1, \\
h_n &= \frac{n! g_n+1}{(n+1)!} = (g_1 q)^m \sum_{j=0}^{\infty} Q_j \frac{q^j}{j!} = (g_1 q)^m \sum_{n=m}^{\infty} Q_n - m \frac{q^n}{(n-m)!},
\end{align*}
\]

which due to [17, 23] give

\[
Q_0 = 1, \quad Q_{n-m} = \sum_{i=1}^{n-m} \binom{m}{i} \ B_{n-m, i}(h_1, h_2, ..., h_{n-i+1}), \quad n - m \geq 1.
\]

Hence by the Eqns. (7) and (8), the coefficients are

\[
\alpha(n, m) = \begin{cases} 
0, & 0 \leq n \leq m - 1, \\
\frac{(g_1 q)^m}{m!}, & n = m, \\
\frac{(g_1 q)^m}{m!} Q_{n-m}, & m + 1 \leq n.
\end{cases}
\]

Therefore (7) - (9) imply (5) and (6).

**Theorem 2.2** For all \(m, r \in \mathbb{Z}_+\), \(\mathbb{Z}_+ = (a set of positive integers)\) and \(x \in (-\infty, \infty)\) define a Fourier series such that

\[
H(m, r; x) = \sum_{n=-\infty}^{\infty} C(n, m, r) e^{inx}, \quad i = \sqrt{-1},
\]

then for any \(r > 0\), and due to (10) there exists a formula

\[
C(r, m, r) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \int_{-\infty}^{\infty} H(m, r; x) e^{-i r x} dx; n = r, r > 0, m > 0 \\
0: n \neq r, r > 0, m > 0
\end{array} \right.
\]

**Proof.** Consider the series (10) and multiply in its both sides by \(e^{-i r x}\), for any fixed \(r > 0, x \in (-\infty, \infty)\), then integrate that sides with respect to \(x\) from \(-\infty\) to \(\infty\) to get

\[
\begin{align*}
\frac{1}{2\pi} \int_{-\infty}^{\infty} H(m, r; x) e^{-i r x} dx \\
= \sum_{n=-\infty}^{\infty} C(n, m, r) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(n-r)x} dx + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(n-r)x} dx \right] \\
+ \sum_{n=-\infty}^{\infty} C(n, m, r) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(n-r)x} dx
\end{align*}
\]

(12)
By the Eqn. (12) and due to the formulae used in [3, 4, 13], we immediately find the result (11).

**Theorem 2.3** *In the series (10) of the Theorem 2.2, if suppose that*

\[
C(n, m, \eta) = \begin{cases} 
  \alpha(n, m), & n \geq m + 1 \\
  \alpha(m, m), & n = m \\
  0, & m - 1 \geq n \geq 0 \\
  0, & n < 0
\end{cases}
\]

*Then \( \forall x \in [0, 2\pi] \), there exists an inequality*

\[
\left| 2m! H(m; x) - \sum_{n=m+1}^{\infty} \frac{2m!(\xi_1)^n}{m!(n-m)!} Q_{n-m} e^{i\eta x}\right| \leq \sqrt{\left( \frac{2m!}{m!} \right)} (\xi_1)^m
\]

(14)

**Proof.** Consider a series under the conditions given in (13) and make an appeal to equalities (9), we write it as

\[
H(m; x) = \sum_{n=-\infty}^{\infty} C(n, m, \eta) e^{i\eta x} = \sum_{n=-\infty}^{0} 0 e^{i\eta x} + \sum_{n=0}^{n-1} 0 e^{i\eta x} + \sum_{n=m}^{\infty} a(n, m) e^{i\eta x} + \sum_{n=m+1}^{\infty} a(n, m) e^{i\eta x}
\]

\[
= a(m, m) e^{i\eta x} + \sum_{n=m+1}^{\infty} a(n, m) e^{i\eta x}
\]

(15)

Now in series (15) use the equalities (9) that implies as

\[
\left| H(m; x) - \sum_{n=m+1}^{\infty} \frac{(\xi_1)^n}{m!(n-m)!} Q_{n-m} e^{i\eta x}\right|^2 \leq \frac{(\xi_1)^{2m}}{m!(n-m)!} |e^{2i\eta x}|
\]

(16)

Finally, \( \forall x \in [0, 2\pi] \) and by the inequality (16), we arrive on the result (14).

**Theorem 2.4** *If all conditions of the Theorem 2.3 are followed, then for all \( x \in (-\infty, \infty) \) and \( m \in \mathbb{Z}^+ \) there exists a formula*

\[
\int_{-\infty}^{\infty} H(m; x) dx = \sum_{n=m}^{\infty} \frac{(\xi_1)^n}{m!(n-m)!} Q_{n-m} \sum_{j=1}^{\eta} \frac{\pi c \exp[-b_1 n]}{\pi b_1} \frac{c}{\pi b_1} \frac{c}{\pi b_1}
\]

(17)

**Proof.** For all \( x \in (-\infty, \infty) \) consider a function

\[
f(x) = \frac{e^{-c/(x^2+b^2)}}{\pi b_1}
\]

and another function \( H(m; x) \) for all \( m \in \mathbb{Z}^+ \) and \( x \in (-\infty, \infty) \) from the Eqn. (15), we get

\[
\int_{-\infty}^{\infty} \frac{f(x)}{\pi b_1} \frac{c}{\pi b_1} \frac{c}{\pi b_1} dx = \sum_{n=m}^{\infty} \frac{(\xi_1)^n}{m!(n-m)!} Q_{n-m} \sum_{j=1}^{\eta} \frac{\pi c \exp[-b_1 n]}{b_1} \frac{c}{\pi b_1} \frac{c}{\pi b_1}
\]

(18)

The Eqn. (18) immediately gives us the formula (17).
It is remarked that on making an appeal to the techniques and results of [12, 28], the formula (17) become very useful in evaluation of various trigonometrically functions and polynomials.

2 Arithmetic functions their partial and complete Bell polynomials and Fourier transformation identities

Various authors to them for example [16-18] studied the arithmetic functions \( r_k(n) \) and \( t_k(n) \), which are the number of representations of \( n \) as a sum of \( k \) squares and \( k \) triangular numbers, respectively. The function \( p_k(n) \) is the number of color partitions of \( n \).

Again, we recall the function due to [1, 23]

\[
F(q) = \prod_{n=1}^{\infty} \frac{1 - q^n}{1 + q^n}\]  

Here in (19) consider that

\[
F^k = \prod_{n=1}^{\infty} \frac{1 - q^n}{1 + q^n} \quad G(q) = - 2 \sum_{n=1}^{\infty} n D(n) q^n. \]  

In the relations (20), \( D(n) \) is the sum of the inverses of the odd divisors of \( n \), that is, \( D(n) = \sum_{odd \; d|n} \frac{1}{d} \), and as \( r_k(n) \) is the number of representations of a positive integer \( n \) as a sum of \( k \) squares, such that representations with different orders and signs are counted as distinct [8, 9, 14, 19]. Therefore following relations are found [23]

\[
f_k(n) = (-1)^n r_k(n), \quad g_n = -2n D(n) = n g_1 D(n), \quad g_1 = -2, \]

(21)

whose application into (4)-(6) gives the expression

\[
r_k(n) = \frac{2^n}{m!} k^n + (-1)^n \sum_{j=1}^{n-1} a(n, j) k^j, \]

such that

\[
a(n, m) = \frac{(-2)^m}{m! (n - m)!} \sum_{j=1}^{n-m} \left( \begin{array}{c} m \\ j \end{array} \right) j! \times B_{n-m,j}(1! D(2), 2! D(3), ..., (n-m-j+1)! D(n-m-j+2)). \]

(22)

Again consider that [11, 23]

\[
F(q) = (q; q)_\infty = \prod_{n=1}^{\infty} (1 - q^n), \quad F^k = \sum_{n=0}^{\infty} p_k(n) q^n, \quad G(q) = - \sum_{n=1}^{\infty} \sigma(n) q^n, \]

(23)

where \( p_k(n) \) is the number of partitions [10, 15] with \( k \) colors (see [16-18]) and \( \sigma(n) \) is the sum of the divisors of \( n \); hence there are the relations (also see in [23])

\[
f_k(n) = p_k(n), \quad g_n = - \sigma(n) = g_1 \sigma(n), \quad g_1 = -1, \]

(25)
and (4)-(6) imply the relation
\[ p_k(n) = \frac{(-1)^n}{n!} k^n + \sum_{j=1}^{n-1} a(n, j) k^j, \]  
(26)
such that
\[ a(n, m) = \frac{(-1)^m}{m! (n-m)!} \sum_{j=1}^{n-m} \binom{m}{j} j! B_{n-m, j} \left( \frac{1}{2} \sigma(2), \frac{2}{3} \sigma(3), ..., \frac{(n-m-j+1)!}{(n-m-j+2)!} \right). \]
(27)

Further consider that
\[ F(q) = \sum_{n=0}^\infty q^{\frac{n(n+1)}{2}} = \prod_{n=1}^\infty \frac{1 - q^{2n}}{1 - q^n} = \prod_{n=1}^\infty (1 + q^n)^2 (1 - q^n), \]  
(28)
And
\[ F_k = \sum_{n=0}^\infty t_k(n) q^n, \quad \quad G(q) = -\sum_{n=1}^\infty n T(n) q^n, \]  
(29)
where \( t_k(n) \) is the number of representations of \( n \) as the sum of \( k \) triangular numbers, such that representations with different orders are counted as unique, and:
\[ T(j) = \sum_{d|j} \frac{1 + 2(-1)^j}{d} = \frac{1}{j} \sum_{d|j} (-1)^d d, \]  
(30)
hence
\[ f_k(n) = t_k(n). \]  
(31)
and from (4)-(6)
\[ t_k(n) = \frac{1}{n!} k^n + \sum_{j=1}^{n-1} a(n, j) k^j, \]  
(32)
where
\[ a(n, m) = \frac{1}{m! (n-m)!} \sum_{j=1}^{n-m} (-1)^j \binom{m}{j} j! B_{n-m, j} \left( \sigma(2), \sigma(3), ..., \frac{(n-m-j+1)!}{(n-m-j+2)!} \right). \]
(33)

Due to the Eqns. (21)-(33), here in our research work, we use following interesting recurrence relations [23] to obtain Fourier series identities
\[ p_{k+1}(n) = \sum_{j=0}^n a(j) p_k(n-j), \]  
(34)
provided that
\[ a(j) = \begin{cases} 
0, & j \neq \frac{m}{2} (3m+1), m = 0, \pm 1, \pm 2, \ldots \\
(-1)^m, & j = \frac{m}{2} (3m+1), m = 0, \pm 1, \pm 2, \ldots 
\end{cases} \]
(35)
\[ r_{k+1}(n) = \sum_{j=0}^n b(j) r_k(n-j), \]  
(36)
provided that
Now we present some identities on Fourier transformations due to arithmetic functions defined in the Eqns. (34) to (39).

The $H_1(x; k, L) = \sum_{n=-\infty}^{\infty} C_{n,k}^{(1)} \exp \left[ \frac{2n \pi i x}{L} \right]$, 

where, $C_{n,k}^{(1)} = \left\{ \begin{array}{ll} p_{k+1}(n), & n \geq 0; \\ 0, & n < 0. \end{array} \right.$

Then in (43) use the definition (35) to obtain

$$\frac{1}{L} \int_{L/2}^{L/2} H_1(x; k, L) e^{-\frac{2\pi n i x}{L}} dx = \sum_{n=0}^{\infty} a(j) \frac{1}{L} \int_{L/2}^{L/2} e^{-\frac{2\pi n j i x}{L}} \left\{ \sum_{n=0}^{\infty} p_k(n) e^{-\frac{2\pi n j i x}{L}} \right\} dx.$$ 

Theorem 3.2 If $H_2(x; k, L) = \sum_{n=-\infty}^{\infty} C_{n,k}^{(2)} \exp \left[ \frac{2n \pi i x}{L} \right]$, 

where, $C_{n,k}^{(2)} = \left\{ \begin{array}{ll} r_{k+1}(n), & n \geq 0; \\ 0, & n < 0. \end{array} \right.$
\[ i = \sqrt{-1}, \ L > 0, \ \forall x \in \mathbb{R} \ \text{a set of real numbers}, \ k \in \mathbb{Z}_+ \ \text{a set of positive integers}, \]

then \( \forall N \geq m^2, \text{where } m \geq 1, \text{there exists following identical Fourier transforms} \)
\[
\frac{1}{L} \int_{-L/2}^{L/2} H_2(x; k, L)e^{-\frac{2\pi N x}{L}} dx = \frac{1}{L} \int_{-L/2}^{L/2} \exp \left[ -\frac{2\pi N x}{L} \right] \left\{ 1 + 2 \exp \left[ \frac{2m^2 i x}{L} \right] \right\} H_2(x; k - 1, L) dx. \tag{47}
\]

**Proof.** Making an appeal to the recurrence relation of the arithmetic function \( r_k(n) \) given in (36) and the definition of Fourier series (45)-(46), we have
\[
H_2(x; k, L) = \sum_{n=0}^{\infty} r_{k+1}(n)e^{\frac{2\pi n x}{L}} = \sum_{n=0}^{\infty} e^{\frac{2\pi n x}{L}} \sum_{j=0}^{\infty} b(j) r_k(n-j) = \sum_{n=0}^{\infty} r_k(n)e^{\frac{2\pi n x}{L}} \sum_{j=0}^{\infty} b(j) e^{\frac{2\pi j x}{L}}
\]
\[ \Rightarrow \frac{1}{L} \int_{-L/2}^{L/2} H_2(x; k, L)e^{\frac{2\pi n x}{L}} dx = \sum_{j=0}^{\infty} b(j) \frac{1}{L} \int_{-L/2}^{L/2} e^{\frac{2\pi (N-j) x}{L}} \left\{ \sum_{n=0}^{\infty} r_k(n)e^{\frac{2\pi n x}{L}} \right\} dx. \tag{48}
\]

Then in (48) the definition (37), we obtain
\[
\frac{1}{L} \int_{-L/2}^{L/2} H_2(x; k, L)e^{\frac{2\pi n x}{L}} dx = \frac{1}{L} \int_{-L/2}^{L/2} \exp \left[ -\frac{2\pi N x}{L} \right] H_2(x; k - 1, L) dx
\]
\[ + \frac{2}{L} \int_{-L/2}^{L/2} \exp \left[ -\frac{2\pi (N-m^2) i x}{L} \right] H_2(x; k - 1, L) dx, \tag{49}\]
provided that \( \forall N \geq m^2; m \in \mathbb{Z}^+, \mathbb{Z}^+ = \{ \text{set of positive integers} \} \).

The Eqn. (49) immediately gives the identity (47).

**Theorem 3.3** If
\[
H_3(x; k, L) = \sum_{n=-\infty}^{\infty} C_{n,k}^{(3)} \exp \left[ \frac{2\pi n x}{L} \right],
\]
where
\[
C_{n,k}^{(3)} = \left\{ \begin{array}{ll}
t_{k+1}(n), & n \geq 0; \\
0, & n < 0.
\end{array} \right.
\]
\[ i = \sqrt{-1}, \ L > 0, \ \forall x \in \mathbb{R} \ \text{a set of real numbers}, \ k \in \mathbb{Z}_+ \ \text{a set of positive integers}, \]

then \( \forall N \geq \frac{m}{2}(m + 1), \text{where } m \in \mathbb{Z}_+ \cup \{0\}, \text{there exists following identical Fourier transforms} \)
\[
\frac{1}{L} \int_{-L/2}^{L/2} H_3(x; k, L)e^{\frac{2\pi n x}{L}} dx = \frac{1}{L} \int_{-L/2}^{L/2} \exp \left[ -\frac{2\pi m(N\frac{m}{2}(m+1)) i x}{L} \right] H_3(x; k - 1, L) dx. \tag{52}\]

**Proof.** Making an appeal to the recurrence relation of the function \( t_k(n) \) in (38) and the definition of Fourier series in (50)-(51), we get
\[
H_3(x; k, L) = \sum_{n=0}^{\infty} t_{k+1}(n)e^{\frac{2\pi n x}{L}} = \sum_{n=0}^{\infty} e^{\frac{2\pi n x}{L}} \sum_{j=0}^{\infty} c(j) t_k(n-j) = \sum_{n=0}^{\infty} t_k(n)e^{\frac{2\pi n x}{L}} \sum_{j=0}^{\infty} c(j) e^{\frac{2\pi j x}{L}}
\]
\[ \Rightarrow \frac{1}{L} \int_{-L/2}^{L/2} H_3(x; k, L)e^{\frac{2\pi n x}{L}} dx = \sum_{j=0}^{\infty} c(j) \frac{1}{L} \int_{-L/2}^{L/2} e^{\frac{2\pi (N-j) x}{L}} \left\{ \sum_{n=0}^{\infty} t_k(n)e^{\frac{2\pi n x}{L}} \right\} dx. \tag{53}\]

Then in (53) the definition (39), we obtain
\[
\frac{1}{L} \int_{-L/2}^{L/2} H_3(x; k, L)e^{\frac{2\pi n x}{L}} dx = \frac{1}{L} \int_{-L/2}^{L/2} \exp \left[ -\frac{2\pi (N\frac{m}{2}(m+1)) i x}{L} \right] H_3(x; k - 1, L) dx,
\]
provided that \( \forall N \geq \frac{m}{2}(m + 1), \text{where } m \in \mathbb{Z}_+ \cup \{0\}. \) (54)

Hence the identity (52) exists.

4 Generalization of the recurrence formula (2) and certain results
Consider the formula (2) in the form

\[ n f_k(n) = \sum_{j=1}^{n} h(j) f_k(n-j), \]  

(55)

and write the Eqn. (55) in the form

\[ y_n = \sum_{j=1}^{n} \binom{n-1}{j-1} x_j y_{n-j}, \]  

(56)

with \( y_n = n! \, f_k(n) \) and \( x_j = (j-1)! \, h(j) \); then (56) implies the following expression

\[ y_n = B_n(x_1, ..., x_n), \]  

(57)

in terms of the complete Bell polynomials [26] and therefore we have

\[ f_k(n) = \frac{1}{n!} \, B_n(0! \, h(1), 1! \, h(2), ..., (n-1)! \, h(n)). \]  

(58)

On the other hand, in [1] were proved the recurrence relations

\[ r_k(n) = -\frac{2k}{n} \sum_{j=1}^{n} (-1)^j j D(j) \, r_k(n-j), \]

\[ t_k(n) = -\frac{k}{n} \sum_{j=1}^{n} j T(j) \, t_k(n-j), \]

\[ p_k(n) = -\frac{k}{n} \sum_{j=1}^{n} \sigma(j) \, p_k(n-j), \]  

(59)

with the structure (2), which is a particular case of (59); then the application of (55) to (59) gives the interesting relations

\[ r_k(n) = \frac{1}{n!} \, B_n(2k \cdot 1! \, D(1), -2k \cdot 2! \, D(2), 2k \cdot 3! \, D(3), ..., -2k(-1)^n \cdot n! \, D(n)), \]

\[ t_k(n) = \frac{1}{n!} \, B_n(-k \cdot 1! \, T(1), -k \cdot 2! \, T(2), -k \cdot 3! \, T(3), ..., -k \cdot n! \, T(n)), \]

\[ p_k(n) = \frac{1}{n!} \, B_n(-k \cdot 0! \, \sigma(1), -k \cdot 1! \, \sigma(2), -k \cdot 2! \, \sigma(3), ..., -k \cdot (n-1)! \, \sigma(n)), \]  

(60)

which are closed expressions with the participation of the complete Bell polynomials. From (60) it is evident that these arithmetical functions are polynomials in \( k \) of degree \( n \), in accordance with (4).

Similarly, the known relation for the partition function [24, 25]

\[ n p(n) = \sum_{j=1}^{n} \sigma(j) \, p(n-j), \]

(61)

with the structure (55)-(57), implies the property

\[ p(n) = \frac{1}{n!} \, B_n(0! \, \sigma(1), 1! \, \sigma(2), 2! \, \sigma(3), ..., (n-1)! \, \sigma(n)), \]  

(62)
in accordance with the result in [19, Theorem 7]. Besides, Robbins [24, 25] deduced the following recurrence relation

\[ n \ p_d(n) = \sum_{j=1}^{n} \sigma_d(j) \ p_d(n-j), \]  

(63)

similar to (61), where \( p_d(n) \) is the number of partitions of \( n \) using only distinct parts and given as

\[ \sigma_d(n) = \sum_{d \mid n} d = \sum_{d \mid n} (-1)^{d-1} \frac{n}{d}; \]  

(64)

then it is immediate the expression

\[ p_d(n) = \frac{1}{n!} B_n(0! \ \sigma_d(1), 1! \ \sigma_d(2), 2! \ \sigma_d(3), \ldots, (n-1)! \ \sigma_d(n)). \]  

(65)

Finally, from (63) it is possible to obtain the identity [24-25] as

\[ \sigma_d(n) = \sum_{j=1}^{n} (-1)^{j-1} j \ p_o(j) \ p_d(n-j), \]  

(66)

where \( p_o(n) \) is the number of partitions of \( n \) into parts which are odd and distinct.

5 On a recurrence relation of Apostol [2, 24]

In this section, we recall the Eqns. (1) - (3) to consider the case

\[ F^k = 1 + \sum_{n=1}^{\infty} f_k(n) \ q^n = \prod_{n=1}^{\infty} (1 - q^n)^{\frac{-k}{n}} Q(n), \]  

(67)

we obtain

\[ G(q) = \sum_{n=1}^{\infty} \frac{Q(n)}{1-q^n} \ q^n = \sum_{n=1}^{\infty} (\sum_{d\mid n} Q(d)) \ q^n, \]  

(68)

therefore (2) implies the recurrence relation of Apostol [2, 24]

\[ n \ f_k(n) = k \ \sum_{j=1}^{n} (\sum_{d\mid n} Q(d)) \ f_k(n-j). \]  

(69)

If \( Q(n) = -n \), then \( f_k(n) = p_k(n) \) and \( \sum_{d\mid n} Q(d) = -\sigma(n) \), thus (69) implies (61)

\[ n \ p_k(n) = -k \ \sum_{j=1}^{n} \sigma(j) \ p_k(n-j), \]  

which was originally found by Gandhi [7, 15], and for \( k = -1 \) it gives (63), and if \( k = 1 \) it generates the identity obtained by Robbins [24] and Osler-Hassen-Chandrupatla [20]

\[ \sigma(n) = -n \ a(n) - \sum_{j=1}^{n-1} \sigma(j) \ a(n-j) \quad n \geq 2, \]  

(70)

with the \( a(j) \) defined in (35).

In [23] there is also a study of polynomial expression (4) and its coefficients.

6 Concluding remarks

The Fourier series is the summation of trigonometrically functions found in the
literature and applicable in various scientific problems for example see in [3, 4, 13]. We observe that the recurrence relations obtained by our own techniques on applying Bell polynomials become helpful to express the Fourier transformation identities obtained in the Section 3.

References


