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A Note On $G(N, \epsilon, \alpha)C_1$ Summability of Derived Series of Fourier Series

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Abstract: In this paper, we introduce a new definition of $G(N,\varepsilon,\alpha)C_1$ summability and present a theorem which establishes conditions for the G(N, p, q) C_1 summability of the derived series of a Fourier series. We provide the example to validate the theorem and deduce corollaries based on it. Overall, this paper contributes to the understanding and analysis of summability and derived series in the context of Fourier series.
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1. Introduction

The summability of Fourier series and its derived series has been a subject of significant research in the field of mathematical analysis. The concept of summability provides a way to assign meaningful values to divergent series, allowing for the approximation of functions through truncated series. One approach to summability is through the use of product summability methods, such as the G(N,p,q)C_1 summability. Quite good amount of works are known on product summability means of the form $(C,1)(E,1)$ of Fourier series and its allied series. Our work builds upon previous research in the field of summability theory (see, Bhatt & Kathal [1996], Borwein [1958], Dikshit [1969], Hardy [1913], Lal & Verma [1998], Prasad [1981], Sharma [1969], [1970], Sinha & Kumar [2007], Sinha and Singh [1993], and so on) which contributed to the understanding of product summability methods, Fourier series, and the properties of derived series.

In this paper, we focus on establishing a theorem regarding the $G(N,p,q)C_1$ means (Product summability) of the derived series of a Fourier series. The derived series, which captures the behavior of the derivatives of the original series, plays a crucial role in understanding the convergence properties of the Fourier series and its approximations. Our theorem provides conditions under which the derived series of a Fourier series is G (N, p, q) C_1 summable to the derivative of the function being approximated. The proof of our theorem involves leveraging lemmas and integrating by parts to establish the convergence of the derived series. We provide detailed steps and explanations to validate our claims. Additionally, we derive corollaries from the theorem, presenting alternative conditions under which the derived series can be summable.

In the following sections, we present the detailed definitions, theorems, and proofs to support our main theorem. Through our research, we aim to contribute to the body of knowledge in summability theory and provide valuable insights into the convergence properties of derived series.

2. **Prelimnaries**

To develop our theorem, we begin by introducing relevant definitions and notations related to summability theory. We define the convolution operation $(\varepsilon*\alpha)_n$ for two given sequences $\{\varepsilon\}_n$ and $\{\alpha\}_n$, which allows us to combine two sequences, and the Nörlund transform, a generalized transform used in summability theory. We also discuss the concept of absolute summability using the notation G(N, p, α) with an index σ, which provides a measure of convergence for series. The convolution operation is commutative and associative. We explore the (C, 1) mean of a series $\sum u_n$ denoted by σ_n and its transformation, G(N, ε, α)C_1. Furthermore, we examine the Fourier series of a periodic function f(t) and introduce the derived series. Notations such as ϕ x (t) and ψx (t) are defined for further analysis.

Definition (Sinha And Singh (1993). Let $\{p_n\}$ is a sequence then

$\Delta\varepsilon_n = p_n - p_{n-1} = \Delta p_n$

Given two sequence $\{\varepsilon\}_n$ and $\{\alpha\}_n$, the convolution $(\varepsilon * \alpha)_n$ defined as,

$$(\varepsilon * \alpha)_n = \sum_{v=0}^{n} p_v \alpha_v \qquad (1.1)$$

operation convolution is commutative and associative, we note that

$$\sum_{v=0}^{n} \varepsilon_v = (\varepsilon * 1)_n \qquad (1.2)$$

$$\Delta(\varepsilon * \alpha)_n = (\Delta\varepsilon * \alpha)_n = (\varepsilon * \Delta\alpha)_n \qquad (1.3)$$

$$N_n^{\varepsilon,\alpha}(s) = \frac{(\varepsilon * \alpha s)_n}{(\varepsilon * \alpha)_n} = \frac{1}{(\varepsilon * \alpha)_n}\sum_{v=0}^{n} \varepsilon_{n-v}\alpha_v s_v \qquad (1.4)$$

$$T_n^{\varepsilon,\alpha}(s) = \frac{(\Delta\varepsilon * \alpha s)_n}{(\Delta\varepsilon * \alpha)_n} = \frac{(\varepsilon * \Delta\alpha s)_n}{(\varepsilon * \Delta\alpha)_n} = \frac{1}{(\varepsilon * \Delta\alpha)_n}\sum_{v=0}^{n} \varepsilon_{n-v}(\alpha_v s_v - \alpha_{v-1}s_{v-1}). \qquad (1.5)$$

Definition . A series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable $G(N,p,\alpha)$ with index σ if

$$\sum_{n=1}^{\infty} n^{\sigma-1}\left|N_n^{(\sigma)} - N_{n-1}^{(\sigma)}\right| < \infty \qquad (1.6)$$

where

$$N_n^{(\sigma)} \equiv \left(N_n^{(\varepsilon,\alpha)}\right)$$

Let $\sum u_n$ be an infinite series whose n-th partial sum is denoted by $\{S_n\}$, write

$$\sigma_n = \frac{S_0 + S_1 + S_2 + S_3 + \dots\dots\dots\dots + S_n}{n+1} \qquad (1.7)$$

$= (C,1)$ mean of the series $\sum u_n$ or sequence $\{s_n\}$.

For any two sequence $\{\varepsilon_n\},\{\alpha_n\}$ of real numbers such that $\varepsilon_0 > 0.\ \alpha_0 > 0$ and we write

$$t_n^{\varepsilon,\alpha} = \frac{1}{(\varepsilon * \alpha)_n} \sum_{k=0}^{n} \varepsilon_{n-k} \alpha_k S_k$$

Where $(\varepsilon * \alpha)_n = \sum_{k=0}^{n} \varepsilon_{n-k}\alpha_k \qquad (\neq 0 \text{ for all } n)$

The generalized Nörlund transform ($G(N,\varepsilon,\alpha)$ transform) of the sequence $\{s_n\}$ is the sequence $\{t_n^{\varepsilon,\alpha}\}$. If $t_n^{\varepsilon,\alpha} \to G$ as $n \to \infty$ then the sequence $\{s_n\}$ is said to be summable by generalised Nörlund method $G(N,\varepsilon,\alpha)$ to S and is denoted by (BORWEIN [1958])

$S_n = S\{G(N,\varepsilon,\alpha)\}$.

The $G(N,\varepsilon,\alpha)$ transform of the $(C,1)$ transform C_1 difines $G(N,\varepsilon,\alpha)C_1$ transform of the partial sum $\{S_n\}$ of the series $\sum u_n$. Thus, if

$$t_n^{\varepsilon,\alpha,c_1} = \frac{1}{(\varepsilon * \alpha)_n} \sum_{k=0}^{n} \varepsilon_{n-k} \alpha_k \sigma_{n-k} \qquad (1.8)$$

tends to S, as $n \to \infty$ then the series $\sum u_n$ is said to be summable by $G(N,\varepsilon,\alpha)C_1$ mean or $G(N,\varepsilon,\alpha)C_1$ summable to S. It is denoted as

$t_n^{\varepsilon,\alpha,c_1} \to G((N,\varepsilon,\alpha)C_1)$.

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $(-\pi,\pi)$. The Fourier series of $f(t)$ is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (1.9)$$

The derived series of the Fourier series is

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \qquad (1.10)$$

We shall use the following notations

$$\phi_x(t) = f(x+t) - f(x-t) - 2f(x) \qquad (1.11)$$
$$\psi_x(t) = f(x+t) - f(x-t). \qquad (1.12)$$

$$G(NC)_1^{\varepsilon,\alpha} = \sum\nolimits_{n^{k=0}}^{n} \frac{\varepsilon_{n-k}\alpha_k}{(n-k+1)} \cdot \frac{\sin^2 (n-k+1)\frac{1}{2}t}{\sin^2 \left(\frac{1}{2}\right) t} \qquad (1.13)$$

$(\varepsilon * 1)_1 = (\varepsilon * 1)(\mathrm{t})$
$\varepsilon_1 = \varepsilon(\mathrm{t})$
$\alpha t = \alpha(t)$.

3. Main Result

We prove the theorem for $\mathrm{G(N.p,q)C_1}$ summability (product summability) as our main result.
We need the following lemmas for the proof of our theorem.

***Lemma* 1 :**

For $0 < x \leq \frac{1}{n}$

$\left|\mathrm{G(NC)}_1^{\varepsilon,\alpha}\right| = \mathrm{O}(\mathrm{n})$

Proof:

By (1.13)

$$\left|G(NC)_1^{\varepsilon,\alpha}\right| = \frac{1}{\pi(\varepsilon*\alpha)_n}\left|\sum_{k=0}^{n}\frac{\varepsilon_{n-k}\alpha_k}{(n-k+1)}\cdot\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\left(\frac{t}{2}\right)}\right|$$

Expanding $\sin(n-k+1)\frac{t}{2}$ in power of $\sin\left(\frac{t}{2}\right)$.

$$\leq \frac{1}{\pi(\varepsilon*\alpha)_n}\sum_{k=0}^{n}\frac{\varepsilon_{n-k}\alpha_k}{(n-k+1)}(n-k+1)^2\frac{\left|\sin^2\frac{t}{2}\right|}{\left|\sin^2\frac{t}{2}\right|} = \frac{1}{\pi(\varepsilon*\alpha)_n}\sum_{k=0}^{n}\varepsilon_{n-k}\alpha_k(n-k+1)$$

$$\leq \frac{2n}{\pi}\left(\frac{1}{(\varepsilon*\alpha)_{k=0}}\sum_{k=0}^{n}\varepsilon_{n-k}\alpha_k\right)(\because n+1\leq 2n) = O(n).$$

***Lemma* 2:**

For $\frac{1}{n} < x \leq \delta < \pi$

$$\left|G(NC)_1^{\varepsilon,\alpha}\right| = O\left(\frac{(\varepsilon*1)_n\alpha_n}{nt^2(\varepsilon*1)_n}\right)$$

Proof : By (1.13)

$$\left|G(NC)_1^{\varepsilon,\alpha}\right| = \frac{1}{\pi(\varepsilon*\alpha)_n}\left|\sum_{k=0}^{n}\frac{\varepsilon_{n-k}\alpha_k}{(n-k+1)}\cdot\frac{\sin^2(n-k+1)\frac{t}{2}|}{\sin^2\left(\frac{t}{2}\right)t}\right|$$

$$= \frac{1}{2\pi(\varepsilon*\alpha)_n}\sum_{k=0}^{n}\frac{\varepsilon_{n-k}\alpha_k \quad |1-\cos(n-k+1)t}{(n-k+1) \qquad \sin^2\left(\frac{1}{2}\right)t}$$

$$\leq \frac{1}{\pi(\varepsilon*\alpha)_n}\sum_{k=0}^{n}\frac{\varepsilon_{n-k}\alpha_k}{(n-k+1)}\frac{1}{\sin^2\left(\frac{1}{2}\right)t}$$

$$\leq \frac{1}{\pi(\varepsilon*\alpha)_n}\sum_{k=0}^{n}\frac{\varepsilon_{n-k}\alpha_k}{(n-k+1)}\frac{\pi^2}{t^2}$$

$$= \frac{\pi}{(\varepsilon*\alpha)_n}\cdot\frac{\alpha_n}{t^2}\sum_{k=0}^{n}\frac{\varepsilon_{n-k}}{n-k+1}\theta, \quad \text{since } \sin\theta \geq \frac{2\theta}{n}$$

$$= \frac{\pi}{(\varepsilon*\alpha)_n}\cdot\frac{\alpha_n}{t^2}\left(\frac{(\varepsilon*1)_n}{n}\right)$$

$$= O\left(\frac{(\varepsilon*1)_n}{nt^2(\varepsilon*\alpha)_n}\right)$$

Theorem A : If for a given point x, the function $\psi_x(t)$ is absolutely continous on $(0,\pi)$ and the Schwarz derivative $f^{'}(x) = \psi_x^{'}(0)$ exists finitely and

$$\int_0^t \left|\psi_x(t) - f^{'}(x)\right| dt = O\left(\frac{t\beta\left(\frac{1}{t}\right)}{(\varepsilon * 1)\left(\frac{1}{t}\right)}\right), \text{ as } t \to +0 \qquad (3.1)$$

then the derived series of a Fourier series (1.10) is $G(N, \varepsilon, \alpha)C_1$ summable to $f^{'}(x)$ at the point $t = x$, provided $\{\varepsilon_n\}$ is non-negative monotonic non-increasing sequence, $\{\alpha_n\}$ is a non-negative monotonic non-decreasing sequence and $\beta(t)$ is a positive nonincreasing function of t such that

$$\beta(n)\alpha_n = O((\varepsilon * \alpha)_n), \text{ as } n \to \infty \qquad (3.2)$$

And
$$\sum_{k=0}^{n} \frac{\varepsilon_n}{k+1} = O\left(\frac{(\varepsilon*1)_n}{n}\right) \qquad (3.3)$$

Proof. Denoting by $S_m(x)$ the sum of the first m-th terms of the series (6.1.10) at the point $t = x$, we have

$$S_m(x) = \frac{1}{\pi}\sum_{n=0}^{m} n\left\{\cos nx \int_{-\pi}^{\pi} f(u)\sin nu du - \sin nx \int_{-\pi}^{\pi} f(u)\cos nu du\right\}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\left[\sum_{n=0}^{m} \text{n}\sin n(u-x)\right] du$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [f(x+t) - f(x-t)]\left[\sum_{n=0}^{m} \text{n}\sin nt\right] dt$$

$$= \frac{-1}{\pi}\int_{0}^{\pi} \psi_x(t)\left[\frac{d}{dt}\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right] dt.$$

Since $\psi_x(t)$ is absolutely continuous on $(0,\pi)$, integration by parts gives

$$S_m(x) = \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\left(\frac{1}{2}\right)t}\psi_x^{'}(t)dt \left(\because \psi_x(\pi) = \psi_x(0) = 0\right)$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\left(\frac{1}{2}\right)}\left(\psi_x^{'}(t) - f^{'}(x)\right)dt + f^{'}(x).$$

By using (1.7), we have

$$\sigma_n(x) - f'(x) = \frac{1}{\pi(n+1)} \int_0^{\pi} \sum_{m=0}^{n} \frac{\sin\left(m + 1\frac{1}{1}\right) t}{\sin\left(\frac{1}{2}\right) t} \left(\psi_x'(t) - f'(x)\right) dt$$

$$= \frac{1}{\pi(n+1)} \int_0^{\pi} \frac{\sin^2 (n+1)\frac{1}{2}}{\sin^2 \left(\frac{1}{2}\right) t} (\psi_x'(t) - f'(x)) dt$$

or

$$\sigma_{n-k}(x) - f'(x) = \frac{1}{\pi(n-k+1)} \int_0^{\pi} \frac{\sin^2 (n-k+1)\frac{t}{2}\pi}{\sin^2 \left(\frac{1}{2}\right) t} (\psi_x'(t) - f'(x) dt$$

so using (1.8), we get

$$t_n^{\varepsilon,\alpha,c_1} - f'(x) = \int_0^{\pi} \left[\frac{1}{\pi(\varepsilon*\alpha)_n} \sum_{k=0}^{n} \frac{\varepsilon_{n-k}\alpha_k}{n-k+1} \frac{\sin^2 (n-k+1)\frac{t}{2}t}{\sin^2\left(\frac{t}{2}\right)\tau}\right] \left(\psi_x'(t) - f'(x)\right) dt$$

$$= \int_0^{\pi} G(NC)_1^{\varepsilon,\alpha} \left(\psi_x'(t) - f'(x)\right) dt$$

$$= \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right] G(NC)_1^{\varepsilon,\alpha} \left(\psi_x'(t) - f'(x)\right) dt$$

$$= I_1 + I_2 + I_3;$$

First of all, let us consider I_1, using Lemma 1, we have

$$I_1 \le \int_0^{\frac{1}{n}} \left|G(NC)_1^{\varepsilon,\alpha} \| \psi_x'(t) - f'(x)\right| dt$$

$$= O\left[n \int_0^{\frac{1}{n}} \left|\psi_x'(t) - f'(x)\right| dt\right]$$

$$= O\left[nO\left(\frac{\frac{1}{n}\beta(n)}{(\varepsilon*1)(n)}\right)\right] \text{ by} \qquad (3.1)$$

$$= O\left[\frac{\beta(n)}{(\varepsilon*1)(n)}\right]$$

$$= O(1) \text{ as } n \to \infty \text{ by hypothesis of the theorem.}$$

Next by using Lemma 2 and integrating by parts, we have

$$I_2 \le \int_{\frac{1}{n}}^{\delta} \left|G(NC)_1^{\varepsilon,\alpha} \| \psi_x'(t) - f'(x)\right| dt$$

$$= \int_{\frac{1}{n}}^{\delta} O\left(\frac{(\varepsilon*1)_n \alpha_n}{nt^2(\varepsilon*\alpha)_n}\right) \left|\psi_x'(t) - f'(x)\right| dt$$

$$= O\left(\frac{(\varepsilon*1)_n\alpha_n}{n(\varepsilon*\alpha)_n}\right)\left[\left(\frac{1}{t^2}O\left(\frac{t\beta\left(\frac{1}{t}\right)}{(\varepsilon*1)\left(\frac{1}{t}\right)}\right)\right)_{\frac{1}{n}}^{\delta} + \right.$$

$$\left. + \int_{\frac{1}{n}}^{\delta} \frac{1}{t^3}O\left(\frac{t\beta\left(\frac{1}{t}\right)}{(\varepsilon*1)\left(\frac{1}{t}\right)}\right)dt\right], \text{ by } (3.1)$$

$$= \left[\left(\frac{(\varepsilon*1)_n\alpha_n}{n(\varepsilon*\alpha)_n}\right)\left(\frac{n^2\beta(n)}{n(\varepsilon*1)(n)}\right)\right] +$$

$$+O\left[\left(\frac{(\varepsilon*1)_n\alpha_n}{n(\varepsilon*\alpha)_n}\right)_{\frac{1}{n}}^{\delta}\int_{t^2}\frac{1}{t^2}\frac{t\left(\frac{1}{t}\right)}{(\varepsilon*1)\left(\frac{1}{t}\right)}dt\right]$$

$$= \left[\frac{\alpha_n\beta(n)}{(\varepsilon*\alpha)_n}\right] + O\left[\left(\frac{(\varepsilon*1)_n\alpha_n}{n(\varepsilon*\alpha)_n}\right)_{\frac{1}{\delta}}^{n}\int_{(\varepsilon*1)(u)}^{(u)} dt\right]$$

$\left(\text{take}\,\frac{1}{t} = u\right)$

$$= \left[\frac{\alpha_n\beta(n)}{\pi(\varepsilon*\alpha)_n}\right] + O\left[\left(\frac{(\varepsilon*1)_n\alpha_n}{n(\varepsilon*\alpha)_n}\right)\frac{\beta(n)n}{(\varepsilon*1)(n)}\right]$$

$\left(\because \frac{\beta(n)n}{(\varepsilon*1)(n)} \text{ is monotonic}\right)$

$$= \left[\frac{\alpha_n\beta(n)}{(\varepsilon*\alpha)_n}\right] + O\left[\frac{\alpha_n\beta(n)}{(\varepsilon*\alpha)_n}\right]$$

$$= O(1) + O(1) \text{ by } (7.3.2)$$

$$= O(1)$$

Lastly, by Riemann-Lebesgue theorem and regularity condition of summability $G(N,\varepsilon,\alpha)C_1$, we have

$$I_3 \le \int_{\delta}^{\pi} \left|G(NC)_1^{\varepsilon,\alpha} \parallel \psi_x'(t) - f'(x)\right| dt$$

$= O(1)$ as $n \to \infty$

Thus from (3.1), (3.2), (3.3) and (3.4) we get

$t_n^{\varepsilon,\alpha,c_1} - f'(x) = O(1);$ as $n \to \infty$

This completes the proof of the theorem.

We deduce some corollaries from our theorem.

Corollary 1: If the condition

$$\int_0^t \left|\psi_x^{'}(t) - f^{'}(x)\right| dt = 0\left(\frac{t}{\log\frac{1}{t}}\right), \text{ as } t \to +0$$

holds then the derived series of the Fourier series is $G(N,p,q)C_1$ summable to $f^{'}(x)$ at the point $t = x$.

Corollary 2 : If the condition

$$\int_0^t \left|\psi_0^{'}(t) - f^{'}(x)\right| dt = O(t), \text{ as } t \to +0$$

holds then the derived series of the Fourier series is $G(N,p,q)C_1$ summable to $f^{'}(x)$ at the point $t = x$. Proof of the corollaries (i) and (ii) can be obtained parallel to the theorem.

Conclusion

The main focus of the paper is to prove a theorem regarding the $G(N,\varepsilon,\alpha)C_1$ summability (product summability) of the derived series of a Fourier series. The theorem states conditions under which the derived series of a Fourier series is G(N,ε,α)C_1 summable to the derivative of the function being approximated. The corollaries derived from the theorem provide additional conditions under which the derived series can be summable.

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