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More Results On *K-d*-Frames

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Abstract: In the present paper, we discuss about some more properties of ***K-d***-frames and their duals in $\mathcal{H}$. We extend the concept of stability of the ***K***-frame and tight ***K***- frame under some perturbations for the double sequences. We also generalize the concept of quotient of bounded operator of ***K***-frame. Furthermore, we show the relationship between the optimal bounds of ***K***-frames and their duals.

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Introduction

Frames were first introduced by Duffin and Schaeffer (1952) to determine and to solve some non-harmonic series problems. After a few decades, Daubechies et al (1986) extended the concept of frames to find series expansions of functions in a set of all square integrable functions over field $\mathbb{R}(\mathcal{L}^2(\mathbb{R}))$. Frames can have infinitely many representations of vectors after removing the uniqueness property from bases in a Hilbert spaces, therefore redundancy becomes the vital property of frames which makes frames far applicable than bases. Due to its flexibility, there are several applications of frames in a variety of fields of mathematics and engineering such as signal and image processing (Ferreira 1999), filter bank theory (Bolcskei et al 1998), harmonic analysis (Grochenig 2001), wireless communications (Heath and Paulraj 2002) and there are many more to observe. For more review on literature one may refer to Benedetto and Fickus (2003), Casazza (2000,2003) and Christensen (2003).

As yet several generalizations of frames have been presented with many applications. Some of those have been introduced such as fusion frames (Casazza and Kutyniok 2004), continuous fusion frames (Faroughi and Ahmadi 2010), generalized frames (Xiao et al 2015), K-frames (Ramu and Johnson 2016) and d-frames (Biswas et al 2023) etc.

Throughout this paper, $\mathcal{H}$ denotes Hilbert/separable Hilbert space, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ a collection of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ (If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, then it is denoted by $\mathcal{B}(\mathcal{H})$). For $K \in \mathcal{B}(\mathcal{H})$, $R(K)$ is the range space of K. K^* is an adjoint of K and $K^{\dagger}$ is a pseudo- inverse of K.

Recentl, Biswas et al. (2023) gave a new generalization of frame with the help of double sequences using the fact that every Bessel sequence in a Hilbert space need not to be a frame.
Biswas et al. (op cit) gave the following definition of d- frame.

Definition 1.1. (Biswas et al. 2023)

A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in $\mathcal{H}$ is said to be a d-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left|\langle x, x_{ij}\rangle\right|^2 \le B\|x\|^2, \ \forall\, x \in \mathcal{H}, \qquad (1.1)$$

here, constants A and B are called lower and upper d- frame bounds respectively. If $A = B$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called tight d- frame. If $A = B = 1$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called Parseval d- frame.

If only the right hand inequality holds in equation (1.1), then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a double Bessel sequence or d-Bessel sequence for $\mathcal{H}$.

In a different manner, Gavruta (2012) proposed the notion of K- frame to study atomic systems in a separable Hilbert space. Gavruta (op cit) gave the following definition of K- frame.

Definition 1.2. (Gavruta op cit)
Let $\mathcal{H}$ be a separable Hilbert space and $K \in \mathcal{B}(\mathcal{H})$. A sequence $\{x_n\}_{n=1}^{\infty}$ is called K- frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \le \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \le B\|x\|^2, \ \forall\, x \in \mathcal{H},$$

here, constants A and B are called lower and upper K-frame bounds respectively.

It is notable that K-frames are more general than ordinary frames (Jia and Zhu 2018, Xiao et al 2013).

Considering this fact, we generalized the concept of frames generated by double sequences with the help of linear bounded operator K and introduced K-d-frames in our previous paper (Pauriyal and Joshi Op cit).
Infact Pauriyal and Joshi (Op cit) gave the following definition of K-d-frame.

Definition 1.3. (Pauriyal and Joshi Op cit)
Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a double sequence in separable Hilbert space $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$. Then, $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a K-d-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \le \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left|\langle x, x_{ij}\rangle\right|^2 \le B\|x\|^2, \ \forall\, x \in \mathcal{H}, \qquad (1.2)$$

here, constants A and B are called lower and upper K-d-frame bounds respectively.

(i) If $A\|K^*x\|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left|\langle x, x_{ij}\rangle\right|^2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a tight K-d-frame.

(ii) If $A = 1$, the above equality becomes $\|K^*x\|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \left|\langle x, x_{ij}\rangle\right|^2$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called Parseval K- d-frame.

Remark 1.4. If only the right hand inequality holds in equation (1.2), then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is called a K-d- Bessel sequence for $\mathcal{H}$.

Remark 1.5. For $K = I$, K-d- frames are d- frames.

Remark 1.6. Every K-frame is a K-d-frame.

Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in separable Hilbert space $\mathcal{H}$ is a K-d- frame, so it is a K-d- Bessel sequence. So, we define the operators, $T : l^2(\mathbb{N}\times\mathbb{N}) \to \mathcal{H}$ by

$$T\left(\{a_{ij}\}_{i,j\in\mathbb{N}}\right) = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} a_{ij}x_{ij}\ ,\ \forall \{a_{ij}\}_{i,j\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N})$$

and $T^* : \mathcal{H} \to l^2(\mathbb{N}\times\mathbb{N})$ by

$$T^*x = \{\langle x, x_{ij}\rangle\}_{i,j\in\mathbb{N}},\ \forall\ x \in \mathcal{H}.$$

Then, $S = TT^*$ be a frame operator from $\mathcal{H} \to \mathcal{H}$ such that

$$Sx = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} \langle x, x_{ij}\rangle x_{ij}\ ,\ \forall\ x \in \mathcal{H}.$$

Main Results

We use the following definition for the further study of paper.

Definition 2.1. (Casazza 2000)
Let $\mathcal{H}$ be a Hilbert space, and suppose that $K \in \mathcal{B}(\mathcal{H})$ has a closed range. Then, there exists a pseudo-inverse $K^\dagger \in \mathcal{B}(\mathcal{H})$ such that

$$N(K^\dagger) = R(K)^\perp,\ R(K)^\dagger = N(K^\perp),\ KK^\dagger = I,$$

and it is uniquely determined for all $x \in R(K)$. In fact, if K is invertible, then $K^{-1} = K^\dagger$.

Corollary 2.2. (Pauriyal and Joshi op cit)
Let $K_1 \in \mathcal{B}(\mathcal{H}_1)$ and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K_1-d-frame for $\mathcal{H}_1$. Let $K_2 \in \mathcal{B}(\mathcal{H}_2)$ and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be surjective with $TK_1 = K_2T$. Then, $\{T x_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d- frame for $\mathcal{H}_2$.

Now, we extend and prove the existing results on K-frame (Jia and Zhu 2008, He et al 2019, Neyshaburi et al 2019 and Xia et al 2013) for the K-d-frame.

Theorem 2.3. Let $K_1 \in \mathcal{B}(\mathcal{H}_1)$ and let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a tight K_1-d-frame for $\mathcal{H}_1$. Let $K_2 \in \mathcal{B}(\mathcal{H}_2)$ be injective with a closed range and let $L \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with $LK_1 = K_2L$. Then $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d- frame for $\mathcal{H}_2$ if and only if L is surjective.

Proof. Let L is surjective then by corollary 2.2 $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d- frame for $\mathcal{H}_2$.
For the converse part, let $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d- frame for $\mathcal{H}_2$ with bounds A_1 and B_1. Then,

$$A_1\,\|K_2^*y\|^2 \le \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} |\langle y, Lx_{ij}\rangle|^2 \le B_1\|y\|^2, \forall y \in \mathcal{H}_2. \tag{2.1}$$

Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a tight K_1-d-frame for $\mathcal{H}_1$ with frame bound A, so

$$A\|K_1^*x\|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2$$

$$A\|K_1^* L^* y\|^2 \le \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} |\langle L^*y, x_{ij}\rangle|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n} |\langle y, Lx_{ij}\rangle|^2$$

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$$\leq B\|L^*y\|^2 \leq B\|L\|^2\|y\|^2.$$

Since, $LK_1 = K_2L$. So, $K_1^*L^* = L^*K_2^*$.

We know that $L \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ has a closed range $R(K_2)^* \subset R(L)$, then from the definition 2.1, L has the pseudo-inverse $L^\dagger$ such that $LL^\dagger = I$. This implies $(L^\dagger)^* L^* = I$.

Then, for all $x \in R(L)$

$\|x\| = \|(L^\dagger)^*L^*x\| \leq \|L^\dagger\| \|L^*x\|$

implies

$\|L^\dagger\|^{-1}\|x\| \leq \|L^*\|\|x\|, \ x \in R(L).$

Now

$$A\|K_1^* L^*y\|^2 = A\|L^*K_2^* y\|^2 \geq A\|L^\dagger\|^{-2} \|K_2^* y\|^2.$$

For all $y \in \mathcal{H}_2$,

$$A\|L^\dagger\|^{-2} \|K_2^* y\|^2 \leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle y, Lx_{ij}\rangle|^2 \leq B\|L\|^2\|y\|^2, \ y \in \mathcal{H}_2.$$

Hence, $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ is a K_2-d- frame for $\mathcal{H}_2$.

We give the following result to address the stability of K-d- frame.

Theorem 2.4. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d- frame for separable Hilbert space $\mathcal{H}$ and let $\{\alpha_{ij}\}_{i,j\in\mathbb{N}}$ and $\{\beta_{ij}\}_{i,j\in\mathbb{N}} \subset \mathbb{R}$ be two positively confined double sequences. Let $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a d-Bessel sequence and there exist constants $0 \leq \lambda, \mu \leq \frac{1}{2}$ such that

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \alpha_{ij}x_{ij} - \beta_{ij}y_{ij}\rangle|^2 \leq \lambda \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \alpha_{ij}x_{ij}\rangle|^2 + \mu \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \beta_{ij}y_{ij}\rangle|^2, \ \forall x \in \mathcal{H},$$

then $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d- frame for $\mathcal{H}$.

Proof. We have,

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \beta_{ij}y_{ij}\rangle|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \beta_{ij}y_{ij} - \alpha_{ij}x_{ij} + \alpha_{ij}x_{ij}\rangle|^2$$

$$\leq 2\left(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \beta_{ij}y_{ij} - \alpha_{ij}x_{ij}\rangle|^2 + \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \alpha_{ij}x_{ij}\rangle|^2\right)$$

$$\leq 2\left(\mu \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \beta_{ij}y_{ij}\rangle|^2 + \lambda \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \alpha_{ij}x_{ij}\rangle|^2 + \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \alpha_{ij}x_{ij}\rangle|^2\right).$$

$$(1 - 2\mu) \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \beta_{ij}y_{ij}\rangle|^2 \leq 2(1 + \lambda) \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, \alpha_{ij}x_{ij}\rangle|^2$$

$$(1-2\mu)\left(\inf_{i,j\in\mathbb{N}}\beta_{ij}\right)^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2 \le 2(1+\lambda)\left(\sup_{i,j\in\mathbb{N}}\alpha_{ij}\right)^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2$$

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2 \le \frac{2(1+\lambda)\left(\sup_{i,j\in\mathbb{N}}\alpha_{ij}\right)^2}{(1-2\mu)\left(\inf_{i,j\in\mathbb{N}}\beta_{ij}\right)^2} B\|x\|^2 .$$

Similarly,

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \alpha_{ij}x_{ij}\rangle\right|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \alpha_{ij}x_{ij}-\beta_{ij}y_{ij}+\beta_{ij}y_{ij}\rangle\right|^2$$

$$\le 2\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \alpha_{ij}x_{ij}-\beta_{ij}y_{ij}\rangle\right|^2 + \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \beta_{ij}y_{ij}\rangle\right|^2\right)$$

$$\le 2\left(\lambda\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \alpha_{ij}x_{ij}\rangle\right|^2 + \mu\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \beta_{ij}y_{ij}\rangle\right|^2\right)$$

$$+\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \beta_{ij}y_{ij}\rangle\right|^2$$

$$(1-2\lambda)\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \alpha_{ij}x_{ij}\rangle\right|^2 \le (2\mu+1)\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \beta_{ij}y_{ij}\rangle\right|^2$$

$$\frac{(1-2\lambda)}{(2\mu+1)}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \alpha_{ij}x_{ij}\rangle\right|^2 \le \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, \beta_{ij}y_{ij}\rangle\right|^2$$

$$\frac{(1-2\lambda)\left(\inf_{i,j\in\mathbb{N}}\alpha_{ij}\right)^2}{(2\mu+1)\left(\sup_{i,j\in\mathbb{N}}\beta_{ij}\right)^2}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2 \le \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2$$

$$\frac{(1-2\lambda)\left(\inf_{i,j\in\mathbb{N}}\alpha_{ij}\right)^2}{(2\mu+1)\left(\sup_{i,j\in\mathbb{N}}\beta_{ij}\right)^2} A\|K^*x\|^2 \le \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2 .$$

Hence, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d- frame for $\mathcal{H}$.

We give the following result on stability of perturbation for K-d- frame.

Theorem 2.5. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d- frame for separable Hilbert space $\mathcal{H}$ with bounds A, B and $\lambda, \mu, \nu \in [0,\infty)$ such that $\max\{\lambda+\nu\sqrt{A^{-1}}\,\|K^{\dagger}\|, \mu\} < 1$. If $\{y_{ij}\}_{i,j\in\mathbb{N}} \subset \mathcal{H}$ be a double Bessel sequence and satisfies

$$\left\|\sum_{r,t=1}^{m,n} c_{rt}(x_{rt}-y_{rt})\right\| \le \lambda\left\|\sum_{r,t=1}^{m,n} c_{rt}x_{rt}\right\| + \mu\left\|\sum_{r,t=1}^{m,n} c_{rt}x_{rt}\right\| + \nu\left(\sum_{r,t=1}^{m,n}|c_{rt}|^2\right)^{\frac{1}{2}}, \forall c_{rt}, r,t \in \mathbb{N}. \quad (2.2)$$

Then $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a $P_{Q(R(K))}\,K$-d- frame for $\mathcal{H}$ with frame bounds $\frac{\left[\sqrt{A}\|K^{\dagger}\|^{-1}(1-\lambda)-\nu\right]^2}{(1+\mu)^2\|K\|^2}$, $\frac{\left[\sqrt{B}(1+\lambda)+\nu\right]^2}{(1-\mu)^2}$ where, $P_{Q(R(K))}$ is the orthogonal projection operator from $\mathcal{H}$ to $Q(R(K))$, $Q = UT^*$. T, U are the synthesis operators for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ respectively.

Proof. Let T, U are the synthesis operators for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ respectively, then

$$T : l^2\,(\mathbb{N}\times\mathbb{N}) \to \mathcal{H},\; T\alpha = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}x_{ij}$$

$$U : l^2\,(\mathbb{N}\times\mathbb{N}) \to \mathcal{H},\; U\alpha = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\alpha_{ij}y_{ij},$$

where $\alpha = \{\alpha_{ij}\}_{i,j\in N} \in l^2\,(\mathbb{N}\times\mathbb{N})$, from equation (2.2)

$$\|T\alpha - U\alpha\| \le \lambda\|T\alpha\| + \mu\|U\alpha\| + \nu\|\alpha\|_{l^2(\mathbb{N}\times\mathbb{N})},\; \alpha\in l^2\,(\mathbb{N}\times\mathbb{N}). \tag{2.3}$$

Now let $\alpha_x = T^*x \in l^2\,(\mathbb{N}\times\mathbb{N})$, $x\in R(K)$, from equation (2.3)

$$\|Sx - UT^*x\| = \|TT^*x - UT^*x\|$$
$$\le \lambda\|Sx\| + \mu\|UT^*x\| + \nu\|T^*x\|_{l^2(\mathbb{N}\times\mathbb{N})}, \tag{2.4}$$

We have the following equation from Theorem 2.4 of (Pauriyal and Joshi op cit)

$$A\,\|K^{\dagger}\|^{-2}\|x\| \le \|Sx\| = \left\|\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x, x_{ij}\rangle x_{ij}\right\| \le B\|x\|,\;\forall x\in R(K), \tag{2.5}$$

From equation (2.5), we have

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle Sx, x\rangle \le \|Sx\|\|x\| \le A^{-1}\|K^{\dagger}\|^2\|Sx\|^2,\;\forall x\in R(K). \tag{2.6}$$

On combining (2.4) and (2.6)

$$\|Sx - UT^*x\| \le (\lambda + \nu\sqrt{A^{-1}}\|K^{\dagger}\|)\,\|Sx\| + \mu\|UT^*x\|,\;\forall x\in R(K) \tag{2.7}$$

by triangle inequality,

$$\frac{1-(\lambda+\nu\sqrt{A^{-1}}\|K^{\dagger}\|)}{(1+\mu)}\|Sx\| \le \|UT^*x\| \le \frac{1+\lambda+\nu\sqrt{A^{-1}}\|K^{\dagger}\|}{1-\mu}\;\|Sx\|. \tag{2.8}$$

On combining (2.5) and (2.8)

$$\frac{1-(\lambda+\nu\sqrt{A^{-1}}\|K^{\dagger}\|)A\|K^{\dagger}\|^{-2}}{1+\mu}\|x\| \le \|UT^*x\| \le \frac{\left[1+\lambda+\nu\sqrt{A^{-1}}\|K^{\dagger}\|\right]}{1-\mu}B\|x\|. \tag{2.9}$$

Suppose $Q = TT^*$, now we have to prove that $R(Q)$ is closed. For all $\{y_{ij}\}_{i,j\in\mathbb{N}} \subset R(Q)$ satisfying $\lim_{m,n\to\infty} y_{ij} = y,\, y\in\mathcal{H}$ then there exists $x_{ij} \in R(K)$ such that

$$y_{ij} = Q(x_{ij}) \tag{2.10}$$

from (2.9) and (2.10)

$$\|x_{ij+pq} - x_{ij}\| \le C^{-1}\|Q(x_{ij+pq} - x_{ij})\| \le C^{-1}\|(y_{ij+pq} - y_{ij})\|, \tag{2.11}$$

where $C = \frac{1-(\lambda+\nu\sqrt{A^{-1}\|K^{\dagger}\|})A\|K^{\dagger}\|^{-2}}{(1+\mu)}$, it follows that $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is a Cauchy sequence, since $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a Cauchy sequence, so there exists $x \in R(K)$ such that $\lim_{m,n\to\infty} x_{ij} = x$. By the continuity of Q, we have

$$\lim_{m,n\to\infty} y_{ij} = y = \lim_{m,n\to\infty} Q(x_{ij}) = Q(x) \in R(K),$$

this implies $R(Q)$ ia closed. From (2.9) we know that Q is injective on $R(K)$, so we conclude that $Q: R(K) \to R(Q)$ is invertible. On combining (2.8) and (2.9), we have $\forall y \in Q(R(K))$.

$$\|SQ^{-1}(y)\| \le \frac{1+\mu}{1-\left(\lambda+\nu\sqrt{A^{-1}\|K^{\dagger}\|}\right)}\|y\| \quad (2.12)$$

$$\|Q^{-1}(y)\| \le \frac{1+\mu}{[1-(\lambda+\nu\sqrt{A^{-1}\|K^{\dagger}\|})]A\|K^{\dagger}\|^{-2}}\|y\|$$

and also $\forall x \in \mathcal{H}$, we have

$$P_{Q(R(K))}x = QQ^{-1}P_{Q(R(K))}x = U(T^*Q^{-1}P_{Q(R(K))}x)$$
$$= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left(T^*Q^{-1}P_{Q(R(K))}x\right)_{ij} y_{ij}$$

$$\|K^*\left(P_{Q(R(K))}\right)^* x\| = \sup_{y\in\mathcal{H},\|y\|=1} |\langle K^*\left(P_{Q(R(K))}\right)^* x, y\rangle|$$
$$= \sup_{y\in\mathcal{H},\|y\|=1} |\langle x, P_{Q(R(K))}Ky\rangle|$$
$$= \sup_{y\in\mathcal{H},\|y\|=1} \left|\left\langle x, \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left(T^*Q^{-1}P_{Q(R(K))}Ky\right)_{ij} y_{ij}\right\rangle\right|$$
$$= \sup_{y\in\mathcal{H},\|y\|=1} \left|\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\overline{\left(T^*Q^{-1}P_{Q(R(K))}Ky\right)_{ij}}\langle x, y_{ij}\rangle\right|$$
$$\le \sup_{y\in\mathcal{H},\|y\|=1}\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\left(T^*Q^{-1}P_{Q(R(K))}Ky\right)_{ij}\right|^2\right)^{\frac{1}{2}}\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, y_{ij}\rangle|^2\right)^{\frac{1}{2}}$$
$$= \sup_{y\in\mathcal{H},\|y\|=1}\|T^*Q^{-1}P_{Q(R(K))}Ky\|_{l^2(\mathbb{N}\times\mathbb{N})}\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, y_{ij}\rangle|^2\right)^{\frac{1}{2}}$$
$$= \sup_{y\in\mathcal{H},\|y\|=1}\left(\left\langle SQ^{-1}P_{Q(R(K))}Ky, Q^{-1}P_{Q(R(K))}Ky\right\rangle\right)^{\frac{1}{2}}\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, y_{ij}\rangle|^2\right)^{\frac{1}{2}}$$
$$\le \sup_{y\in\mathcal{H},\|y\|=1}\|SQ^{-1}P_{Q(R(K))}Ky\|^{\frac{1}{2}}\|Q^{-1}P_{Q(R(K))}Ky\|^{\frac{1}{2}}\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, y_{ij}\rangle|^2\right)^{\frac{1}{2}}$$
$$\le \frac{1+\mu}{[1-(\lambda+\nu\sqrt{A^{-1}}\|K^{\dagger}\|)]\sqrt{A}\,\|K^{\dagger}\|^{-1}}\|K\|\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, y_{ij}\rangle|^2\right)^{\frac{1}{2}}$$

$$= \frac{(1 + \mu)}{[\sqrt{A}\|K^{\dagger}\|^{-1}(1-\lambda) - \nu]} \|K\| \left(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2 \right)^{\frac{1}{2}}.$$

Hence we have

$$\frac{[\sqrt{A}\|K^{\dagger}\|^{-1}(1-\lambda) - \nu]^2}{(1 + \mu)^2\|K\|^2} \|K^* \left(P_{Q(R(K))}\right)^* x\|^2 \le \left(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2 \right)^{\frac{1}{2}}, \forall x \in \mathcal{H}.$$

Remark 2.6. If $K = I$, then Theorem 2.5 is just perturbation of d-frame.

Corollary 2.7. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d- frame for $\mathcal{H}$ with bounds A and B and $K \in \mathcal{B}(\mathcal{H})$ with closed range and there exists $0 < G < A$. If $\{y_{ij}\}_{i,j\in\mathbb{N}} \subset \mathcal{H}$ be a double Bessel sequence and satisfies

$$\left\| \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} c_{ij}(x_{ij} - y_{ij}) \right\| \le \sqrt{G} \left(\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |c_{ij}|^2 \right)^{\frac{1}{2}}, \forall c_{ij} \in l^2(\mathbb{N}\times\mathbb{N})$$

then $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a $P_{Q(R(K))}$ K-d- frame for $\mathcal{H}$ with frame bounds $\frac{(\sqrt{A}\|K^{\dagger}\|^{-1} - \sqrt{G})^2}{\|K\|^2}$, $(\sqrt{B} + \sqrt{G})^2$, where $P_{Q(R(K))}$ is the orthogonal projection operator from $\mathcal{H}$ to $Q(R(K))$, $Q = UT^*$. T, U are the synthesis operators for $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ respectively.

Proof. On taking $\lambda = \mu = 0$ and $\nu = \sqrt{G}$ in Theorem 2.5.

In the following result we show the condition that make $\{y_{ij}\}_{i,j\in\mathbb{N}}$ a double Bessel sequence or a K-d- frame with the help of a K-d- frame, also we can say this is the relation between a K-d frame and a d- Bessel sequence.

Theorem 2.8. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d- frame for $\mathcal{H}$ with bounds A and B, let $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Then $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a double Bessel sequence for $\mathcal{H}$ if there exists a λ such that

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij} - y_{ij}\rangle|^2 \le \lambda \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2, \forall x \in \mathcal{H}.$$

Moreover, let T_X be the analysis operator of $\{x_{ij}\}_{i,j\in\mathbb{N}}$, then $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d- frame if $\lambda \le \frac{\|T_X^{\dagger}\|^{-2}}{2A}$. On the other side, if $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a double Bessel sequence for $\mathcal{H}$ with T_Y and there exists $a > 0$ such that $\|T_Y\| \le a\|K^*x\|$. Then, we have

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij} - y_{ij}\rangle|^2 \le \lambda \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, x_{ij}\rangle|^2, \forall x \in \mathcal{H}.$$

Proof. Let $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a double Bessel sequence for $\mathcal{H}$ with T_Y, we obtain

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2 = \|T_Y x\|^2 \le a\|K^* x\|^2 \le \frac{a}{A}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2, \forall\, x \in \mathcal{H}.$$

So,

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij} - y_{ij}\rangle\right|^2 \le 2\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2 + \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2\right)$$

$$\le 2\left(\frac{a}{A}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2 + \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2\right)$$

$$= 2\left(\frac{a}{A} + 1\right)\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2$$

$$= \lambda \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2$$

where $\lambda = 2\left(\frac{a}{A} + 1\right)$. The other part is

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij} - x_{ij}\rangle + \langle x, x_{ij}\rangle\right|^2$$

$$\le 2\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij} - x_{ij}\rangle\right|^2 + \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2\right)$$

$$\le 2(\lambda + 1)\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2$$

$$\le 2B(\lambda + 1)\,\|x\|^2.$$

Thus, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be d-Bessel sequence for $\mathcal{H}$. Moreover, we know that there exists an operator $T_X^{\dagger}$ such that $T_X T_X^{\dagger} x = x$, hence

$$\|x\|^2 = \|T_X T_X^{\dagger} x\|^2 \le \|T_X^{\dagger}\|^2 \|T_X x\|^2 = \|T_X^{\dagger}\|^2 \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2.$$

So,

$$\left\|T_X^{\dagger}\right\|^{-2}\|x\|^2 \le \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2,$$

since

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2 = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij} - y_{ij}\rangle + \langle x, y_{ij}\rangle\right|^2$$

$$\le 2\left(\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij} - y_{ij}\rangle\right|^2 + \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2\right).$$

We have

$$\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, y_{ij}\rangle\right|^2 \ge \frac{1}{2}\lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2 - \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij} - y_{ij}\rangle\right|^2$$

$$\geq \frac{1}{2}\left\|T_X^{\dagger}\right\|^{-2}\|x\|^2 - \lambda \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, x_{ij}\rangle\right|^2$$
$$\geq \frac{1}{2}\left(\|T_X^{\dagger}\|^{-2} - 2\lambda A\right)\|x\|^2 .$$

So if $\|T_X^{\dagger}\|^{-2} - 2\lambda A > 0$, that is if $\lambda < \frac{\|T_X^{\dagger}\|^{-2}}{2A}$, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a K-d- frame for $\mathcal{H}$.

We generalize the concept on quotient of bounded operator of K- frame. This result is the extended version of Theorem 5.3 of Ramu and Johnson (2016).
Let $E, F \in \mathcal{B}(\mathcal{H})$, the quotient $[E/F] : R(F) \to R(E)$ defined by $Fx \to Ex$. We note that $Q = [E/F]$ is a linear operator on $\mathcal{H}$ if and only if $N(F) \subset N(E)$. In this case $D(Q) = R(F)$, $R(Q) \subset R(E)$ and $QF = E$. The quotient $[E/F]$ is called a semiclosed operator and its collection is closed under sum and product. We introduce the following result on K-d- frames using the concept of quotients of bounded operators.

Theorem 2.9. Let $\{x\}_{i,j\in\mathbb{N}}$ be a K-d- frame for $\mathcal{H}$ with frame operators S and $L \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:
1. $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ be a LK-d- frame.
2. $[(LK)^*/S^{\frac{1}{2}}L^*]$ is bounded.
3. $[(LK)^*/(LSL^*)^{\frac{1}{2}}]$ is bounded.

Proof. (1) ⇒ (2),

let $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ be a LK-d- frame, then there exists $A > 0$ such that

$$A\|(LK)^*x\|^2 \leq \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle x, Lx_{ij}\rangle\right|^2$$
$$= \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\left|\langle L^*x, x_{ij}\rangle\right|^2$$
$$= \|S^{\frac{1}{2}}(L^*x)\|^2 .$$
Hence, $[(LK)^*/S^{\frac{1}{2}}L^*]$ is bounded.

(2) ⇒ (3),
let $[(LK)^*/S^{\frac{1}{2}}L^*]$ is bounded, then there exists $B > 0$ such that

$$\|(LK)^*x\|^2 \leq B\|S^{\frac{1}{2}}(L^*x)\|^2, \forall x \in \mathcal{H}.$$
So
$$\|(LSL^*)^{\frac{1}{2}}x\|^2 = \left\langle (LSL^*)^{\frac{1}{2}}x, (LSL^*)^{\frac{1}{2}}x\right\rangle$$
$$= \langle (LSL^*)x, x\rangle = \langle SL^*x, L^*x\rangle$$
$$= \|S^{\frac{1}{2}}(L^*x)\|^2$$
$$\geq \frac{1}{B}\|(LK)^*x\|^2 .$$
Hence, $[(LK)^*/(LSL^*)^{\frac{1}{2}}]$ is bounded.

(3) ⇒ (1),

let $[(LK)^*/(LSL^*)^{\frac{1}{2}}]$ is bounded. Then there exists $B > 0$ such that

$$\|(LK)^* x\|^2 \leq B\|(LSL^*)^{\frac{1}{2}} x\|^2, \forall x \in \mathcal{H}.$$

Let

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, Lx_{ij}\rangle|^2 = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle L^* x, x_{ij}\rangle|^2$$

$$= \|S^{\frac{1}{2}}(L^* x)\|^2$$

$$= \left\langle S^{\frac{1}{2}}(L^* x), S^{\frac{1}{2}}(L^* x)\right\rangle$$

$$= \langle (LSL^*)x, x\rangle$$

because LSL^* is positive and self adjoint, its square root exists and it is denoted by $(LSL^*)^{\frac{1}{2}}$.

Hence,

$$\lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, Lx_{ij}\rangle|^2 = \|(LSL^*)^{\frac{1}{2}} x\|^2 \geq \frac{1}{B} \|(LK)^* x\|^2.$$

Hence, $\{Lx_{ij}\}_{i,j\in\mathbb{N}}$ be a LK-d- frame for $\mathcal{H}$.

Corollary 2.10. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a d- frame for $\mathcal{H}$ and let $K \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:

1. $\{Kx_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d- frame for $\mathcal{H}$.
2. $[K^*/S^{\frac{1}{2}}]$ is bounded.

Lemma 2.11. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame and a Bessel sequence $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be K-d-dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$, then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ are K-d-frame and K^* -d-frame respectively.

Proof. We know that

$$Kx = \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, y_{ij}\rangle x_{ij}, \forall x \in \mathcal{H}.$$

$$\|Kx\|^4 = |\langle Kx, Kx\rangle|^2 = \left|\left\langle \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} \langle x, y_{ij}\rangle x_{ij}, Kx\right\rangle\right|^2$$

$$\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2 \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle Kx, x_{ij}\rangle|^2$$

$$\leq \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2 B\|Kx\|^2$$

$$\|Kx\|^2 \leq B \lim_{m,n\to\infty} \sum_{i,j=1}^{m,n} |\langle x, y_{ij}\rangle|^2,$$

where B is an upper bound of $\{x_{ij}\}_{i,j\in\mathbb{N}}$. This implies, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ is a K^* -d-frame. For lower bound, we take

$$K^*x = \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}\langle x, x_{ij}\rangle y_{ij} \ ,\forall\ x\in\mathcal{H}$$

and repeat the above process for K^*.

We give the relation between optimal bounds of K-d-frame and its K-d-dual.

Theorem 2.12. Let K be a closed range operator and let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}$ with optimal bounds A and B respectively and also $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$ with optimal bounds C and D respectively. Then,

$$C \geq \frac{1}{B} \text{ and } D \geq \frac{1}{A}.$$

Proof. Applying Lemma 2.11,

$$\frac{1}{B}\|Kx\|^2 \leq \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, y_{ij}\rangle|^2 \ ,\forall\ x\in\mathcal{H}.$$

This implies

$$C \geq \frac{1}{B}.$$

Similarly

$$\frac{1}{D}\|K^*x\|^2 \leq \lim_{m,n\to\infty}\sum_{i,j=1}^{m,n}|\langle x, x_{ij}\rangle|^2 \ ,\forall\ x\in\mathcal{H}.$$

This implies

$$D \geq \frac{1}{A}.$$

We obtain the following result of equality for K-d-frame and its dual.

Theorem 2.13. Let $K\in\mathcal{B}(\mathcal{H})$ and $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}$ and $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$, then for every bounded sequence of complex numbers $\{z_{ij}\}_{i,j\in\mathbb{N}}$

$$\sum_{i,j\in\mathbb{N}} z_{ij}\,\langle x, y_{ij}\rangle\,\overline{\langle Kx, x_{ij}\rangle} - \left\|\sum_{i,j\in\mathbb{N}} z_{ij}\,\langle x, y_{ij}\rangle\, x_{ij}\right\|^2$$

$$= \left(\sum_{i,j\in\mathbb{N}}(1-\bar{z}_{ij})\,\overline{\langle x, y_{ij}\rangle}\langle Kx, x_{ij}\rangle\right) - \left\|\sum_{i,j\in\mathbb{N}}(1-z_{ij})\langle x, y_{ij}\rangle x_{ij}\right\|^2 \ ,\forall\ x\in\mathcal{H}.$$

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-frame for $\mathcal{H}$, $\{y_{ij}\}_{i,j\in\mathbb{N}}$ be a K-d-dual of $\{x_{ij}\}_{i,j\in\mathbb{N}}$ and consider the operators

$$Ux = \sum_{i,j\in\mathbb{N}} z_{ij}\,\langle x, y_{ij}\rangle x_{ij}\ ,\ Vx = \sum_{i,j\in\mathbb{N}}(1-z_{ij})\langle x, y_{ij}\rangle x_{ij} \ ,\forall\ x\in\mathcal{H},$$

we have $U + V = K$.

$$\sum_{i,j\in\mathbb{N}} z_{ij}\langle x, y_{ij}\rangle \overline{\langle Kx, x_{ij}\rangle} - \left\|\sum_{i,j\in\mathbb{N}} z_{ij}\langle x, y_{ij}\rangle x_{ij}\right\|^2 = \left(\sum_{i,j\in\mathbb{N}} \langle z_{ij}\langle x, y_{ij}\rangle x_{ij}, Kx\rangle\right) - \|Ux\|^2$$

$$= \left(\sum_{i,j\in\mathbb{N}} \langle z_{ij}\langle x, y_{ij}\rangle x_{ij}, Kx\rangle\right) - \langle Ux, Ux\rangle$$

$$= \sum_{i,j\in\mathbb{N}} \langle K^* z_{ij}\langle x, y_{ij}\rangle x_{ij}, x\rangle - \langle U^*Ux, x\rangle$$

$$= \langle K^*Ux, x\rangle - \langle U^*Ux, x\rangle$$

$$= \langle (K^* - U^*)Ux, x\rangle$$

$$= \langle V^*Ux, x\rangle$$

$$= \langle V^*(K - V)x, x\rangle$$

$$= \langle V^*Kx, x\rangle - \langle V^*V x, x\rangle$$

$$= \langle x, K^*V x\rangle - \|Vx\|^2$$

$$= \left(\langle Kx, \sum_{i,j\in\mathbb{N}} (1 - z_{ij})\langle x, y_{ij}\rangle x_{ij}\rangle\right) - \left\|\sum_{i,j\in\mathbb{N}} (1 - z_{ij})\langle x, y_{ij}\rangle x_{ij}\right\|^2$$

$$= \left(\sum_{i,j\in\mathbb{N}} (1 - \bar{z}_{ij})\overline{\langle x, y_{ij}\rangle}\langle Kx, x_{ij}\rangle\right) - \left\|\sum_{i,j\in\mathbb{N}} (1 - z_{ij})\langle x, y_{ij}\rangle x_{ij}\right\|^2.$$

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