J. Mountain Res. P-ISSN: 0974-3030, E-ISSN: 2582-5011 DOI: https://doi.org/10.51220/jmr.v18i2.10
Vol. 18(2), (2023), 79-85

Sasakian surfaces of invariable holomorphic sectional curvature in Hyper-Kaehlerian manifolds

U.S. Negi* • Sulochana • Preeti Chauhan • Neha Rani

Department of Mathematics, H.N.B. Garhwal University (A Central University), S.R.T. Campus Badshahithaul, Tehri Garhwal-249199, Uttarakhand, India.

*Corresponding Author Email id: usnegi7@gmail.com

Received: 11.06.2023; Revised:15.08.2023; Accepted: 10.11.2023

Abstract: Negi and Bisht (2019) measured almost contact hypersurfaces of a Kaehlerian manifold. They (Negi and Bisht 2022) also calculated hypersurface Properties of Almost Contact Kaehlerian Manifolds. In this paper, the author has defined and studied the Sasakian surfaces of invariable holomorphic sectional curvature in Hyper-Kaehlerian manifolds and established several theorems.

Key Words: Holomorphic sectional curvature • Sasakian surfaces • Differentiable manifold • Hyper-Kaehlerian manifolds.

MSC 2010:53C15, 53C50, 53C80, 53B35.

Introduction:

On a (2n −1)-dimensional real differentiable manifold M^{2n-1} with local coordinates system $\{x^i\}$, if there exist a tensor field $\emptyset_j^i$, contravarient and covariant vector field P^i, Q_i and metric tensor g_{jk} satisfying:

$$P^i\, Q_i = 1 \tag{1.1}$$

$$rank\,(\emptyset_j^i) = 2n-2, \tag{1.2}$$

$$\emptyset_j^i\, P^i = 0, \quad \emptyset_j^i Q_i = 0, \tag{1.3}$$

$$\emptyset_j^i\, \emptyset_k^j = -\,\delta_k^i + P^i Q_k\,, \tag{1.4}$$

$$g_{ji} P^i = Q_j\,, \tag{1.5}$$

$$g_{ji}\, \emptyset_k^f\, \emptyset_r^i = g_{kr} - Q_r\, Q_k\,, \tag{1.6}$$

Then the set ($\emptyset_j^i$, P^i, Q_j) is called an almost Sasakian manifold (Sasaki 1960, Gray's 1959). The almost Sasakian manifold with metric g_{ji} is called an almost Sasakian metric manifold. Also, $\eta\wedge\phi\ldots\wedge\phi\ (n-1\ times) = \rho^{-n} d\eta\wedge\ldots\wedge d\eta\ (n-1\ times) \neq 0$ and satisfy a relation (Tashiro 1963):

$$2\rho\phi_{ji} = \partial_j Q_i - \partial_i Q_j, \tag{1.7}$$

Let M^{2n} be 2n-dimensional differentiable manifold in a tensor field F_{λ}^{k} is called Hermitian metric and pair $(F_{\lambda}^{\kappa}, G_{\lambda\kappa})$is an almost Hermitian manifolds and $M^{2n-1}$is defined by $X^{\kappa} = X^{\kappa}(x^{i})$which satisfies:

$$F_{\lambda}^{\mu}F_{\mu}^{\kappa} = -\delta_{\lambda}^{\kappa}, \quad (1.8)$$

$$G_{\kappa\lambda}F_{\mu}^{\kappa}F_{\nu}^{\lambda} = G_{\mu\nu} \quad (1.9)$$

$$\nabla_{\mu}F_{\lambda\kappa} = 0, \quad (1.10)$$

Where ∇ denotes the covariant differentiation in the Hermitian Metric, it is called Kaehlerian manifold.

Assuming that the hypersurfaces M^{2n-1} be Orientable, we put $B_{i}^{\kappa} = \partial_{i}X^{\kappa}$ ($\partial_{i} = \partial/\partial x^{i}$) and **2n** vectors B_{i}^{κ}, C^{κ} dual basis of B_{κ}^{i}, C_{κ} satisfies following relations:

$$B_{i}^{\kappa}B_{\kappa}^{i} = \delta_{i}^{\kappa}, \quad B_{i}^{\kappa}C_{\kappa} = 0, \quad B_{\kappa}^{i}C^{\kappa} = 0, \quad C^{i}C_{\kappa} = 1, B_{i}^{\kappa}B_{\lambda}^{i} + C^{\kappa}C_{\lambda} = \delta_{\kappa}^{\delta}.$$

The Riemannian metric g_{ji} on$M^{2n-1}$is give by $g_{ji} = G_{\lambda\kappa}B_{j}^{\lambda}B_{i}^{\kappa}$. this is called the induced Riemannian metric. The Gauss and Weingarten equations of hypersurface are Fialkow (1938) and Gray (1959):

$$\nabla_{j}B_{i}^{\kappa} = H_{ji}C^{\kappa} \quad \text{and} \quad \nabla_{j}C_{\kappa} = -H_{ji}B_{\kappa}^{i}$$

Where H_{ji} is the second fundamental tensor of the hypersurface.

If the Hermitian manifold $\overline{M}^{2n}$ is Kaehlerian, then we find following two equations given by Tashiro (1963) and Masafumi Okumura (1966)]: then the following identities are satisfied:

$$\nabla_{j}Q_{i} = -\phi_{i}^{r}H_{ji} \quad (1.11)$$

$$\nabla_{j}\phi_{ir} = Q_{i}H_{jr} - Q_{r}H_{ji}. \quad (1.12)$$

$$H_{j}^{r}\phi_{r}^{i} + \phi_{j}^{r}H_{r}^{i} = 2\rho\phi_{j}^{i}. \quad (1.13)$$

Definition (1.1): In a Sasakian surface M^{2n-1} of a Hyper-Kaehlerian manifold, the vector Q^{i} istraitH_{j}^{i} at M^{2n-1}[Masafumi Okumura (1966)]. Therefore, the following properties are satisfied:

$$H_{ji}Q^{j} = \alpha Q_{i}, \quad (\alpha = H_{ji}Q^{j}Q^{i}). \quad (1.14)$$

$$H_{jk} + H_{ri}\phi_{j}^{r} \quad \phi_{k}^{i} - 2\rho g_{jk} - (\alpha - 2\rho)Q_{j}Q_{k} = 0, \quad (1.15)$$

$$H_{r}^{r} = \alpha + 2(n-1)\rho. \quad (1.16)$$

Sasakian surfaces of invariable Holomorphic sectional curvature in Hyper-Kaehlerian manifolds.

Let Y^{n} be a vector in $\overline{M}^{2n}$, then the holomorphic sectional curvature (HSC) with respect to the vector Y^{n} is given by:

$$K = -\frac{\bar{R}_{\nu\mu\lambda\kappa}F_{\kappa}^{\mu}F_{\sigma}^{\kappa}Y^{\nu}Y^{\kappa}Y^{\lambda}Y^{\sigma}}{G_{\lambda\mu}Y^{\lambda}Y^{\mu}G_{\lambda\nu}Y^{\kappa}Y^{\nu}} \quad (2.1)$$

Where $\overline{R}_{\nu\mu\lambda\kappa}$ is covariant component of curvature of $\overline{M}^{2n}$ is invariable HSC and the Kaehlerian manifold of invariable HSC tensor of the form:

$$\overline{R}_{\nu\mu\lambda\kappa} = {}^{K}/_{4}\left(G_{\mu\lambda}G_{\nu\kappa} - G_{\nu\lambda}G_{\mu\kappa} + F_{\mu\lambda}F_{\nu\kappa} - F_{\nu\lambda}F_{\mu\kappa} - 2F_{\nu\mu}F_{\lambda\kappa}\right).$$

Then the Gauss and Colazzi equations are following holds:[Negi and Bisht (2019)],

$$R_{kjir} = B_k^{\nu}B_j^{\mu}B_i^{\lambda}B_r^{k}\overline{R}_{\nu\mu\lambda\kappa} + H_{kr}H_{ji} - H_{jk}H_{ki}, \qquad (2.2)$$

$$\nabla_k H_{ji} - \nabla_j H_{ki} = B_k^{\nu}B_j^{\mu}B_i^{\lambda}C^{k}\overline{R}_{\nu\mu\lambda\kappa}, \qquad (2.3)$$

$$R_{kji} = k\left(g_{ji}g_{kr} - g_{ki}g_{jr} + \phi_{kr}\phi_{ji} - \phi_{jr}\phi_{ki} - 2\phi_{kj}\phi_{ir}\right) + H_{ji}H_{kr} - H_{ki}H_{jr} \qquad (2.4)$$

$$\nabla_k H_{ji} - \nabla_j H_{ki} = k\left(Q_k\phi_{ji} - Q_j\phi_{ki} - 2\phi_{kj}Q_i\right), \qquad (2.5)$$

Then, we get the following identities fulfilled:[Negi and Bisht (2019)],

$$\nabla^r H_{ji} - \nabla_j H_r^r = 0, \qquad (2.6)$$

$$\left(\nabla_k H_{ji} - \nabla_j H_{ki}\right)Q^i = -2k\phi_{kj}, \qquad (2.7)$$

$$\left(\nabla_k H_{ji} - \nabla_j H_{ki}\right)Q^k = k\phi_{ji}, \qquad (2.8)$$

Differentiating (1.14) and use of (1.11), then Sasakian surface in Hyper-Kaehlerian manifold of invariable H-sectional curvature, we find:

$$\nabla_k H_{ji}Q^i - H_j^i\phi_i^r H_{rk} = \nabla_k\alpha Q_j + \alpha\nabla_k Q_j,$$

Then, exterior derivative of a 1-form $\alpha Q = (\alpha Q_i)\, dx^i$ using and obtained:

$$\left(\nabla_k H_{ji} - \nabla_j H_{ki}\right)Q^i - H_j^i\phi_i^r H_{rk} + H_k^i\phi_i^r H_{rj} = \nabla_k\alpha\, Q_j - \nabla_j\alpha\, Q_k + \alpha(\nabla_k Q_j - \nabla_j Q_k),$$

From (1.7) and (2.7), then

$$-2\,k\phi_{ji} - H_j^i\phi_i^r H_{rk} + H_k^i\phi_i^r H_{rj} = \nabla_k\alpha Q_j - \nabla_j\alpha Q_k + 2\,\rho\alpha\phi_{kj}, \qquad (2.9)$$

Transvecting (2.9) with Q^j and taking account of Definition (1.1), we have $\nabla_j\alpha = \beta Q_j, \quad (\beta = Q^r\nabla_r\alpha),$
That is

$$d\alpha = \beta Q, \qquad (2.10)$$

From which $d\beta \wedge Q + \beta dQ = 0,$ or in the components form,

$$\nabla_j\beta Q_i - \nabla_i\beta Q_j + 2\,\rho\beta\phi_{ji} = 0,$$

Because of (1.7). Transvecting this with ϕ^{ji}, we get

$$4(n-1)\,\rho\,\beta = 0$$

And $\rho \neq 0$ therefore α = constant. Then

$$(a\rho + k)\phi_{kj} + H_j^i\phi_i^r H_{rk} = 0, \qquad (2.11)$$

$H_{is}H_j^s - 2\rho H_{ji} + (a\rho + k)g_{ji} + (a\rho - a^2 - k)Q_jQ_i = 0$. (2.12)

This implies that:

$(2\nabla_r\rho - \nabla_r H_i^i)\phi_j^r = 0$, (2.13)

$\nabla_i(H_r^r - 2\rho) = \sigma Q_j$, (2.14)

Ifλ exist a point root of second tensor H_j^i. Also, η^i andv^i be a parallel point roots of λ. That is, from (2.14), we get:

$\lambda^2 - 2c\lambda + ca + k = 0.$ (2.15)

Here, principal curvatures a, λ_1, $\lambda_2$and itsown multiplicities by v_0, v_1, v_2 and V_a, V_{λ_1}, V_{λ_2} then it follows that [Negi and Bisht (2022)]:

$[dimV_\alpha = v_0, dimV_{\lambda_i} = v_i, (i = 1,2)$,

$v_0 + v_1 + v_2 = 2n - 1, T_p(M^{2n-1}) = V_\alpha \oplus V_{\lambda_1} \oplus V_{\lambda_2}]$ (2.16)

and that V_α, V_{λ_1}, andV_{λ_2}, are orthogonal to each other. Also, $P\epsilon M^{2n-1} \Rightarrow T(m^{2n-1})$ denoted D_α and $D_{\lambda_i}(i = 1, 2)$ at P define v_0 *and* v_i $(i = 1, 2)$. [Negi and Bisht (2022)].

Theorem (2.1): For any v^i going to D_{λ_1} (or D_{λ_2}), then vector field $\phi_j^i v^j$ goes to D_{λ_2} (or D_{λ_1}).

Proof: We have equation (2.11) in vector field going to D_{λ_1},

$\phi_r^i H_j^r v^j + H_r^i \phi_j^r v^j = 2c\phi_j^i v^j$.

On the other hand, (2.15) gives that $\lambda_1 + \lambda_2 = 2c$. Consequently $\phi_j^i v^j$ belongs to D_{λ_2}.

Theorem (2.2): Let M^{2n-1} be a Sasakian surface in Hyper-Kaehlerian manifold of invariable sectional curvature and $\dim D_\alpha \geq 2$. Then it is confess two separate principal curvatures.

Proof: Let v^i be a trait vector parallel to principal curvature a. Then from (2.12) inv^j and $H_j^i v^j = av^i$, we find:

$(a^2 - ca + \kappa)(v_i - Q_r v^r Q_i) = 0$.

If v^i is orthogonal to Q^r, we have

$\kappa = ca - a^2$. (2.17)

The algebraic equation distinct roots to a be given by:

$\lambda_1 = c + \sqrt{c^2 - ca - \kappa}$, $\lambda_2 = c - \sqrt{c^2 - ca - k}$. (2.18)

Substituting (2.17) into (2.18), we obtained:

$\lambda_1 = 2c - a, \lambda_2 = a$.

Hence H^i_jconfess on greatest two separate principal curvatures a and $(2c-a)$.From Theorem(2.2), if a Sasakian surface of invariable HSC confess three separate principal curvatures$v_0 = dimD_\alpha = 1$, then we prove the following:

Theorem (2.3): The allocations D_{λ_1} and D_{λ_2} are both integrable.

Proof: Let v^i and u^i be a vector field belongs to D_{λ_1} then we find:

$H^j_i v^i = \lambda_1 v^j, \quad H^j_i u^i = \lambda_i u^j.$

Therefore, for the bracket $[u,v]^i$ of two vectors field u^i and v^i, we have:

$$H^i_j[u,v]^j = H^i_j(u^r \nabla_r v^j - v^r \nabla_r u^j)$$

$$= u^r\{\nabla_r(H^i_j v^j) - \nabla_r H^i_j v^j\} - v^r\{\nabla_r(H^i_j u^j) - \nabla_r H^i_j u^j\}$$

$$= u^r(\lambda_1 \nabla_r v^i - \nabla_r H^i_j v^j) - v^r(\lambda_1 \nabla_r u^i - \nabla_r H^i_j u^j)$$

$$= \lambda_1 [u,v]^i + (\nabla_r H^i_j - \nabla_j H^i_r) v^r u^j.$$

Substituting (2.5) into the above, we get

$$H^i_j[u,v]^j = \lambda_1[u,v]^i + k(Q_r \phi^i_j - Q_j \phi^i_r - 2\phi_{rj} Q^i) v^r u^j.$$

The vectors fields Q^i belonging to D_α and $\phi^i_r v^r$Belonging toD_{λ_α}, we have

$$H^i_j[u,v]^j = \lambda_1[u,v]^i.$$

Which shows that the distribution D_{λ_1}, is integrable. Similarly, we can prove $D_{\lambda_2}$is also integrable and three separate principal curvatures $dimD_{\lambda_1} = dimD_{\lambda_2} = n-1$.

If $\overset{v^i}{(a)}$ be $(n-1) \perp D_{\lambda_1}$, then $\phi^i_r \overset{v^r}{(a)}$ are $(n-1) \perp D_{\lambda_2}$.then H_{ji}symbolize in the following form:

$$H_{ji} = a\eta_j \eta_i + \sum_{a=1}^{n-1} \lambda_1 \overset{v_i}{(a)}\overset{v_j}{(a)} + \sum_{a=1}^{n-1} \lambda_2 \, \phi^r_i \overset{v_r}{(a)} \phi^s_j \overset{v_s}{(a)} \tag{2.19}$$

The allocation D_{λ_2}integrable at all point of P of M^{2n-1},thennbd U of P and (n−1) functions $f^\alpha(a = 1, \dots, n-1)$, then:

$$H^i_j \frac{\partial f^\alpha}{\partial x^i} = \lambda_i \frac{\partial f^\alpha}{\partial x^j},$$

That is,$\frac{\partial f^\alpha}{\partial x^i}$belong to D_{λ_1}.Also, $\overset{v_i}{(a)}$ be incline vectors in the nbd U of *P*.

From (2.19) and use of (2.4), we obtain:

$$(\lambda_2^2 - 2\lambda_1\lambda_2 + a\lambda_1 + k)\phi^s_i \overset{v_s}{(b)} = 0$$

Since $\phi^s_i \overset{v_s}{(b)}$ is non-zero vector, we get:

$$\lambda_2^2 - 2\lambda_1\lambda_2 + a\lambda_1 + k = 0 \qquad (2.20)$$

and

$$P \in Uf^n \frac{\partial f^n}{\partial x^i} H^i_j = \lambda_1 \frac{\partial f^n}{\partial x^j} .M^{2n-1} 2(c^2 - ac - k) = (2c - a)\sqrt{c^2 - ac - k}.$$

Where there are functions f^n satisfying $\frac{\partial f^n}{\partial x^i} H^i_j = \lambda_1 \frac{\partial f^n}{\partial x^j}$. However, λ_1 ,a and k being all invariable, the left hand members of (2.20) are also invariable. Therefore (2.20) is applicable in M^{2n-1}.From (2.18) and (2.20), we get:

$$2(c^2 - ac - k) = (2c - a)\sqrt{c^2 - ac - k} .$$

If $k = c^2 - ac, \Rightarrow \lambda_1 = \lambda_2 = c$. And $k \neq c^2 - ac$ then, we get:

$$k = -\frac{a^2}{4} \qquad (2.21)$$

Theorem (2.4): If Sasakian surfaces of Hyper-Kaehlerian manifold of invariable HSC confess three separate principal curvatures a, λ_1 , λ_2 and two vectors of the integral submanifold of D_{λ_1} is invariable.

Proof. We consider the integral submanifold of D_{λ_1} . Let v^j and w^i be any two commonly orthonormal vectors going to D_{λ_1} .Then curvatures $K'_{(i)}$ w.r.t. the vectors v^i and w^j is given by:

$$K'_{(1)} = -R_{kjir} v^k w^j v^i w^r ,$$

Since the v^i and w^i are unit vectors. Then from (1.5) becomes to:

$$K'_{(1)} = \left\{-k(g_{ji}g_{kr} - g_{ki}g_{jr} + \phi_{ji}\phi_{kr} - \phi_{ki}\phi_{jr} - 2\phi_{kj}\phi_{ir}) - H_{ji}H_{kr} + H_{ki}H_{jr}\right\} v^k w^j v^j w^r$$

Since $\phi^i_j v^i$ belongs to D_λ , we have

$$K'_{(i)} = k + \lambda_1^2, (i = 1, 2),$$

From which, from (2.21), we get:

$$K'_{(1)} = 4c^2 - 2ac, K'_{(2)} = 0, \qquad (2.22)$$

References

Boothby, W. M. and Wang, H.C. (1968). On contact manifolds, Ann. of maths., 68 , 721-734.

Fialkow, A. (1938). Hypersurfaces of a space of constant curvature, Ann. of maths., 39, 762-785.

Gray, J.W. (1959). Some global properties of contact structures, Ann. of Math., 69, 421- 450.

Masafumi Okumura (1966), Contact hypersurfaces in certain Kaehlerian manifolds. Tohoku Math Journ. Vol. 18, No. 1, pp. 74-102.

Negi, U.S. and Bisht Manoj Singh (2019),Some almost contact hypersurfaces of a Kaehlerian manifold., Journal of Interdisciplinary Cycle Research, Volume XI, Issue XII, pp. 1634 -1642,

Negi, U.S. and Bisht Manoj Singh (2022), Hypersurface Properties of Almost Contact Kaehlerian Manifolds, Journal of Mountain Research, Vol. 17(1), pp. 27-32.

Okumura, M. (1964). Certain almost contact hypersurfaces in Kaehlerian manifolds of constant holomorphic sectional curvatures, Tohoku Math. Journal, 16, 270 - 284.

Sasaki, S. (1960), On differentiable manifolds with certain structures which are closely related to almost contact structure, I, Tóhoku Math. Journ., 12, 459-476.

Tashiro, Y. (1963). On contact structure of hypersurfaces in complex manifold, Tóhoku Math. Journ., 15, 62-78.

Yano, K. (1965). Differential geometry on complex and almost complex spaces, pergamon press.

****