Polynomial Expressions for Certain Arithmetic Functions

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Abstract: We exhibit polynomial expressions for the arithmetic functions $r_k(n)$ and $t_k(n)$, the number of representations of n as a sum of k squares and k triangular numbers, respectively, and also for the color partitions $p_k(n)$.

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Introduction

The sums of squares, triangular numbers and divisor sums are used in the representations of the sequential integers.

The sum of divisors is connected to the partitions function (see, Andrews et al. (2022), Berndt (1994), Grosswald (1984, 1985)). On the other hand, sum of divisors function in terms of colour partitions are studied in the researches of various authors (for example, López-Bonilla et al. (2021), López-Bonilla and Morales-García (2022), López-Bonilla and Yaljá Montiel-Pérez (2021)). The following recurrence relation due to Ramanujan is given in (Berndt (1994)) as

(1.1) $$nP(n) = \sum_{i=1}^{n} \sigma(i)P(n-i),$$

where, $\sigma(n)$ be the sum of divisors of n, and $P(n)$ the number of unordered integer partitions of n.

Recently, in (López-Bonilla et al. (2021), Pathan et al. (2022), Shattuck (2017)) the authors have investigated some properties and identities for the (exponential) incomplete Bell polynomials or partial Bell polynomials $B_{n,k}(x_1, \dots, x_{n-k+1})$ through a generating function

(1.2) $$\exp\left[\sum_{i=1}^{\infty} x_i \frac{t^i}{i!}\right] = \sum_{n=k}^{\infty} B_{n,k}(x_1, \dots, x_{n-k+1}) \frac{t^n}{n!}, k \geq 0.$$

Pathan et al. in (2022), the partition function $p(n)$ is written in terms of $Q_m(k)$, the number of partitions of m using (possibly repeated) parts that do not exceed k.

In view of the importance and usefulness of the Ramanujan's sum (1.1), we generalize this result. Further, our aim is to discuss and exhibit polynomial expressions for the arithmetic functions $r_k(n)$ and $t_k(n)$, the number of representations of n as a sum of k squares and k triangular numbers, respectively, and also for the color partitions $p_k(n)$.

In our researches we make an appeal to the following theory and the formulae for exploring new ideas in analytic and arithmetic number theory:

Preliminary theory and formulae used

Recently, Andrews et al (2023) have proved the following theorem:

***Theorem* 2.1.** If $F(q)$ and $G(q)$ are two analytic functions of q for $|q|<1$, with the values $F(0)=1$, $G(0)=0$, such that

(2.1) $q\frac{d}{dq}\log F(q)=G(q),\ \ (F(q))^k=\sum_{n=0}^{\infty} f_k(n)\,q^n\,,\ G(q)=\sum_{n=1}^{\infty} g_n\,q^n\,.$

Then there exists following sequence of functions

(2.2) $f_k(n)=\frac{k}{n}\sum_{j=1}^{n} g_j\ f_k(n-j),$

(2.3) $g_n=-\ n\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}f_k(n)$.

The relation (2.3) is the inversion of (2.2) with respect to the sequence $\{g_n\}$.

It is remarkable that when we set $k=1$ in (2.2), it becomes the result due to Ramanujan (1.1).

We apply Theorem 2.1 on introducing a logarithmic function of the Mittag-Leffler function $E_\alpha(z)$, (see in Mathai and Haubold (2008, p. 80)) defined as

(2.4) $E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\forall\alpha,z\in\mathbb{C},\Re(\alpha)>0.$

and the density integral formula given in (Chandel et al. (1992))

(2.5) $\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-s)x}dx=\begin{cases}1, n=s;\\0, n\neq s.\end{cases}$

In the present paper, our aim is to invert (2.2) for the sequence $\{f_k(n)\}$ by appropriately expressing the coefficients of $\{f_k(n)\}$ involving the partial exponential Bell polynomials (Comtet (1974), Cvijovic (2011), Pathan et al. (2022), Shattuck (2017)). These results are in harmony with the relations of the formulae of Jakimczuk (2022). The Sec. 4 contains applications of such familiar arithmetic functions like $r_k(n)$ and $t_k(n)$, the number of representations of n as a sum of k squares and k triangular numbers, respectively (see in Grosswald (1984, 1985), Moreno, Wagstaff Jr. (2006)), and also $p_k(n)$, the number of color partitions of n (López-Bonilla et al. (2021), López-Bonilla and Morales-García (2022), López-Bonilla and Yaljá Montiel-Pérez (2021)).

Partial exponential Bell polynomials and summation formulae

In this section, we apply the results and formulae (2.1)-(2.3) of the Theorem 2.1. Further using the formulae (2.4) and (2.5), we obtain various summation formulae and the coefficients in terms of partial exponential Bell polynomials (see, Cvijovic (2011), Pathan et al. (2022), Shattuck (2017)).

***Theorem* 3.1** If in the Theorem 2.1, the results (2.1)-(2.3) are followed, then there exists following expressions

(3.1) $a(n,n)=\frac{1}{n!}(g_1)^n,\quad n\geq 1\,,$

and

(3.2) $a(n,m)=\frac{1}{m!\,(n-m)!}\sum_{j=1}^{n-m}(g_1)^{m-j}\binom{m}{j}j!\ B_{n-m,\,j}\left(\frac{1!}{2}g_2\,,\frac{2!}{3}g_3\,,\ldots,\frac{(n-m-j+1)!}{n-m-j+2}g_{n-m-j+2}\right),$
$n\geq m+1,$

where, partial Bell polynomials $B_{n,k}(x_1,\ldots,x_{n-k+1})$ are given by (1.2).

Proof. Consider the results (2.2) and (2.3) and suppose that $f_k(0) = 1,\ k \geq 0,\ f_0(n) = \delta_{0n}\,,\ g_0 = 0\,,$ then formula (2.2) shows that $f_k(n)$ is a polynomial of degree n in k given by

(3.3) $f_k(n) = a(n,n)\, k^n + a(n,n-1)\, k^{n-1} + \cdots + a(n,2)\, k^2 + a(n,1)\, k\,,\ n \geq 1.$

Now, to determine the coefficients $a(n,m)$ in terms of the quantities $g_j,$ we assume that k is a continuous variable in the polynomial (3.3). Then we apply $\frac{d^m}{dk^m}$ to the property $F^k = \sum_{n=0}^{\infty} f_k(n)\, q^n$ and after taking $k = 0,$ we obtain

(3.4) $$\sum_{n=0}^{\infty} m!\ a(n,m) q^n = (\log F)^m = \left(\sum_{n=1}^{\infty} \frac{g_n}{n} q^n\right)^m = (g_1\, q)^m \left(\sum_{n=0}^{\infty} h_n \frac{q^n}{n!}\right)^m,$$

$h_0 = 1,\ h_n = \frac{n!\, g_{n+1}}{(n+1) g_1}\,,$

and then there exists

(3.5) $$(\log F)^m = (g_1\, q)^m \sum_{j=0}^{\infty} Q_j\ \frac{q^j}{j!} = (g_1)^m \sum_{n=m}^{\infty} Q_{n-m}\ \frac{q^n}{(n-m)!}\,.$$

In the results (3.4) and (3.5) apply the formulae of (Pathan et al. (2022), Shattuck (2017)) and find that

(3.6) $$Q_0 = 1, \qquad Q_{n-m} = \sum_{l=1}^{n-m} \binom{m}{l}\ l!\ B_{n-m,l}(h_1\,, h_2\,, \ldots, h_{n-l+1}), \qquad n - m \geq 1,$$

and hence to get

(3.7) $$a(n,m) = \begin{cases} 0, & 0 \leq n \leq m-1, \\ \frac{(g_1)^n}{n!}\,, & n = m, \\ \frac{(g_1)^m}{m!\ (n-m)!}\ Q_{n-m}\,, & m+1 \leq n. \end{cases}$$

Therefore, applying the formulae (3.5)-(3.7) in the results (3.3) and (3.4) we are able to demonstrate the coefficients of the polynomial (3.3) by the following expressions

(3.8) $$a(n,n) = \frac{1}{n!}\,(g_1)^n\,, \quad n \geq 1\,,$$

and

(3.9)
$$a(n,m) = \frac{1}{m!\ (n-m)!}\ \sum_{j=1}^{n-m} (g_1)^{m-j} \binom{m}{j} j!\ B_{n-m,\,j}\left(\frac{1!}{2}\, g_2\,, \frac{2!}{3}\, g_3\,, \ldots, \frac{(n-m-j+1)!}{n-m-j+2}\, g_{n-m-j+2}\right), \quad n \geq m+1.$$

Hence the Theorem 3.1 is followed.

Now by an appeal to the Theorems 2.1 and 3.1 we obtain that:

Theorem 3.2. Consider (2.1) as $G(z) = \sum_{n=0}^{\infty} g_n\, z^n$ and for all $r \in \mathbb{N}^*,\ \mathbb{N}^* = \{2,3,4,\ldots,L\}, L < \infty$ suppose that

(3.10) $$H(z) = \log\left(e^{-z} E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)\right),$$

where $E_\alpha(.),$ is the Mittag-Leffler function $\forall \Re(\alpha) > 0$ defined in (2.4).

Also for all $z \neq 0$ if the relation

(3.11) $z\frac{d}{dz}H(z) = G(z)$ exists,
then following series holds

(3.12) $$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{g_{n-m}}{\Gamma\left(1+\frac{m}{r}\right)} = \sum_{k=1}^{r-1}\frac{1}{\Gamma\left(\frac{k}{r}\right)}.$$

Proof. Consider the function (3.10) due to Pathan and Kumar (2021)

$$H(z) = \log\left(e^{-z}E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)\right),$$

then in it use the formula due to Mathai and Haubold ((2008), p. 84) and make an appeal to (3.11) and find that

(3.13) $$z\frac{d}{dz}H(z) = \frac{\sum_{k=1}^{r-1}\frac{z^{1-\frac{k}{r}}}{\Gamma\left(1-\frac{k}{r}\right)}}{E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)} = G(z).$$

Again, on setting $z = 0$ in (3.10) and (3.13) and then comparing with the Theorem 2.1, we find following initial values

(3.14) $$H(z)\Big|_{z=0} = \log\left(e^{-z}E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)\right)\Big|_{z=0} = 0,\quad F(z)\Big|_{z=0} = e^{-z}E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)\Big|_{z=0} = 1,$$

and

$$G(z)\Big|_{z=0} = \frac{e^{-z}\sum_{k=1}^{r-1}\frac{z^{1-\frac{k}{r}}}{\Gamma\left(1-\frac{k}{r}\right)}}{e^{-z}E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)}\Big|_{z=0} = 0$$

$\Rightarrow H(0) = 0, F(0) = 1, G(0) = 0$ and hence all initial conditions given in (3.14) and of the Theorem 2.1 are followed.

Then making an appeal to the Eqn. (3.13) and the Theorem 2.1, we find

(3.15) $$G(z)E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right) = \sum_{k=1}^{r-1}\frac{z^{1-\frac{k}{r}}}{\Gamma\left(1-\frac{k}{r}\right)}$$

$$\Rightarrow \sum_{n=0}^{\infty} g_n z^n \sum_{m=0}^{\infty}\frac{z^{\frac{m}{r}}}{\Gamma\left(1+\frac{m}{r}\right)} = \sum_{k=1}^{r-1}\frac{z^{1-\frac{k}{r}}}{\Gamma\left(1-\frac{k}{r}\right)}.$$

Again, in left hand side of (3.15) apply the series rearrangement techniques and then in both of the sides of its replace z by $ze^{ix}\forall x \in (-\infty,\infty)$, and thus on multiplying both the sides by e^{-isx}, after integrating both the side with respect to x from $-\infty$ to ∞, we get

(3.16)

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} g_{n-m}\frac{z^{n-m+\frac{m}{r}}}{\Gamma\left(1+\frac{m}{r}\right)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(n-m\left(1-\frac{1}{r}\right)-s\right)x}dx = \sum_{k=1}^{r-1}\frac{z^{1-\frac{k}{r}}}{\Gamma\left(1-\frac{k}{r}\right)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\left(\left(1-\frac{k}{r}\right)-s\right)x}dx$$

$$= \sum_{k=1}^{r-1}\frac{z^{\frac{k}{r}}}{\Gamma\left(\frac{k}{r}\right)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(s-\left(\frac{k}{r}\right)\right)x}dx.$$

Make an appeal to the formula (2.5) in both sides of (3.16) as on putting $s = \frac{k}{r}\forall k = 1,2,3,4,\ldots,r-1; r = 2,3,4.5,\ldots$

and

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$n-m\left(1-\frac{1}{r}\right)-s=0 \Rightarrow n=\left[s+m\left(1-\frac{1}{r}\right)\right]\ \forall m=0,1,2,3,4,\ldots,$ where $[x]$ is the greatest integer less or equal to x, we get

(3.17) $$\sum_{n=0}^{\infty}\sum_{m=0}^{n} g_{n-m}\frac{z^s}{\Gamma\left(1+\frac{m}{r}\right)}=\sum_{k=1}^{r-1}\frac{z^s}{\Gamma\left(\frac{k}{r}\right)}.$$

Therefore on equating like powers of z in both of the sides of (3.17), we get the required result (3.12). Hence the Theorem 3.2 is followed.

Corollary 3.1. If all conditions of Theorem 3.2 are followed, then for all $r=\nu, \nu\in\mathbb{N}^*$, there exists a summation formula

(3.18)
$$\sum_{n=0}^{\infty}\frac{g_{n-0}}{\Gamma(1)}+\sum_{n=0}^{\infty}\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{\nu}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{\nu}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{\nu}\right)}+\ldots+\sum_{n=0}^{\infty}\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{\nu}\right)}=\frac{1}{\Gamma\left(\frac{1}{\nu}\right)}+\frac{1}{\Gamma\left(\frac{2}{\nu}\right)}+\ldots+\frac{1}{\Gamma\left(\frac{\nu-1}{\nu}\right)}.$$

Proof. For different values of r in (3.12), we apply the induction method to obtain the result (3.18).

In the result (3.12) suppose that $r=2$. Thus we find $k=1$ and hence due to (3.15), we have $s=\frac{1}{2}$. Therefore we find a summation formula

(3.19) $$\sum_{n=0}^{\infty}\left\{\frac{g_{n-0}}{\Gamma(1)}+\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{2}\right)}+\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{2}\right)}+\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{2}\right)}+\ldots+\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{2}\right)}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{g_{n-0}}{\Gamma(1)}+\sum_{n=0}^{\infty}\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{2}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{2}\right)}+\ldots+\sum_{n=0}^{\infty}\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{2}\right)}=\frac{1}{\Gamma\left(\frac{1}{2}\right)}.$$

In a similar manner of (3.19) for $r=3,$ we find $k=1,2$ and hence $s=\frac{1}{3},\frac{2}{3}$.
Therefore we have another summation formula

$$\sum_{n=0}^{\infty}\left\{\frac{g_{n-0}}{\Gamma(1)}+\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{3}\right)}+\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{3}\right)}+\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{3}\right)}+\ldots+\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{3}\right)}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{g_{n-0}}{\Gamma(1)}+\sum_{n=0}^{\infty}\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{3}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{3}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{3}\right)}+\ldots+\sum_{n=0}^{\infty}\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{3}\right)}=\frac{1}{\Gamma\left(\frac{1}{3}\right)}+\frac{1}{\Gamma\left(\frac{2}{3}\right)}.$$

Further when $r=4,$ we find $k=1,2,3$ and hence $s=\frac{1}{4},\frac{2}{4},\frac{3}{4}$. Therefore we find another summation formula

$$\sum_{n=0}^{\infty}\left\{\frac{g_{n-0}}{\Gamma(1)}+\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{4}\right)}+\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{4}\right)}+\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{4}\right)}+\ldots+\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{4}\right)}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{g_{n-0}}{\Gamma(1)}+\sum_{n=0}^{\infty}\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{4}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{4}\right)}+\sum_{n=0}^{\infty}\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{4}\right)}+\ldots+\sum_{n=0}^{\infty}\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{4}\right)}=\frac{1}{\Gamma\left(\frac{1}{4}\right)}+\frac{1}{\Gamma\left(\frac{2}{4}\right)}+\frac{1}{\Gamma\left(\frac{3}{4}\right)}.$$

So on when $r=\nu, \nu\in\mathbb{N}^*,$ we find $k=1,2,3,\nu-1$ and hence $s=\frac{1}{\nu},\frac{2}{\nu},\frac{3}{\nu},\ldots,\frac{\nu-1}{\nu}$.
Therefore, finally we find a summation formula

(3.20) $$\sum_{n=0}^{\infty}\left\{\frac{g_{n-0}}{\Gamma(1)}+\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{\nu}\right)}+\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{\nu}\right)}+\frac{g_{n-3}}{\Gamma\left(1+\frac{3}{\nu}\right)}+\ldots+\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{\nu}\right)}\right\}=\frac{1}{\Gamma\left(\frac{1}{\nu}\right)}+\frac{1}{\Gamma\left(\frac{2}{\nu}\right)}+\ldots+\frac{1}{\Gamma\left(\frac{\nu-1}{\nu}\right)}.$$

By Eqn. (3.20), we evaluate the required result (3.18).

Corollary 3.2. If all conditions of the Theorem 3.2 and the corollary 3.1 are followed, then for fixed $r=2,$ there exists a summation formula

(3.21) $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_{n,m}=\frac{1}{\sqrt{\pi}},\ \text{where}\quad A_{n,m}=\begin{cases}\frac{g_{n-m}}{\Gamma\left(1+\frac{m}{2}\right)}, & n\le m;\\ 0, & n>m.\end{cases}$$

Proof. Consider (3.19) for $r=2, m=0$. We obtain that

(3.22) $\sum_{n=0}^{\infty}\frac{g_{n-0}}{\Gamma(1)}=\frac{g_{0-0}}{\Gamma(1)}+\frac{g_{1-0}}{\Gamma(1)}+\ldots+\frac{g_{n-0}}{\Gamma(1)}=\frac{g_{0-0}}{\Gamma(1)}+0+\ldots+0=\frac{g_{0-0}}{\Gamma(1)}, A_{n,m}=0, \forall n\geq 1, m=0.$

Again for $r=2, m=1$, we have

(3.23) $\sum_{n=0}^{\infty}\frac{g_{n-1}}{\Gamma\left(1+\frac{1}{2}\right)}=\frac{g_{0-1}}{\Gamma\left(1+\frac{1}{2}\right)}+\frac{g_{1-1}}{\Gamma\left(1+\frac{1}{2}\right)}+0+\ldots+0=\frac{g_{0-1}}{\Gamma\left(1+\frac{1}{2}\right)}+\frac{g_{1-1}}{\Gamma\left(1+\frac{1}{2}\right)}, A_{n,m}=0, \forall n\geq 2, m=1.$

For $r=2, m=2$, we have

(3.24) $\sum_{n=0}^{\infty}\frac{g_{n-2}}{\Gamma\left(1+\frac{2}{2}\right)}=\frac{g_{0-2}}{\Gamma\left(1+\frac{2}{2}\right)}+\frac{g_{1-2}}{\Gamma\left(1+\frac{2}{2}\right)}+0+\ldots+0=\frac{g_{0-2}}{\Gamma\left(1+\frac{2}{2}\right)}+\frac{g_{1-2}}{\Gamma\left(1+\frac{2}{2}\right)}, A_{n,m}=0, \forall n\geq 2, m=2.$

Similarly, for $r=2, m=n$, we have

(3.25) $\sum_{n=0}^{\infty}\frac{g_{n-n}}{\Gamma\left(1+\frac{n}{2}\right)}=\frac{g_{0-0}}{\Gamma(1)}+\frac{g_{1-1}}{\Gamma\left(1+\frac{1}{2}\right)}+0+\ldots+0=\frac{g_{0-0}}{\Gamma(1)}+\frac{g_{1-1}}{\Gamma\left(1+\frac{1}{2}\right)}, A_{n,m}=0, \forall n\geq 2, m=n.$

Then by an appeal of the results from (3.22) to (3.25), we obtain the result (3.21).

Theorem 3.3. $\forall M=1,2,3,\ldots,\ r\in\mathbb{N}^*$, if all conditions of the Theorem 3.2 are satisfied, then there exists following result

(3.26) $\{e^{z}G(z)\}^{M}=\left[1+\sum_{k=1}^{r-1}\frac{\gamma\left(1-\frac{k}{r},z\right)}{\Gamma\left(1-\frac{k}{r}\right)}\right]^{-M}\sum_{k_1+\ldots+k_{r-1}\leq M}\frac{M!}{k_1!\ldots k_{r-1}!}\left(\frac{z^{\frac{1}{r}}}{\Gamma\left(\frac{1}{r}\right)}\right)^{k_1}\cdots\left(\frac{z^{\frac{r-1}{r}}}{\Gamma\left(\frac{r-1}{r}\right)}\right)^{k_{r-1}},$

where, $\gamma(\alpha,z)=\int_0^z e^{-u}u^{\alpha-1}\,du$ defines the incomplete gamma function.

Proof. Make an appeal to the Eqn. (3.15) of the Theorem 3.2, we write

(3.27) $\{G(z)\}^{M}=\left\{E_{\frac{1}{r}}\left(z^{\frac{1}{r}}\right)\right\}^{-M}\left\{\frac{z^{\frac{1}{r}}}{\Gamma\left(\frac{1}{r}\right)}+\ldots+\frac{z^{\frac{r-1}{r}}}{\Gamma\left(\frac{r-1}{r}\right)}\right\}^{M}.$

Now in the right-hand side of (3.27) apply the formulae due to (Srivastava and Manocha (1984, p. 87)) and to (Mathai and Haubold (2008, Eqn. (2.2.11), p. 84)), we obtain the required formula (3.26).

Polynomials associated to $r_k(n)$, $p_k(n)$ and $t_k(n)$

In this section, we find applications of some of the familiar arithmetic functions $r_k(n)$ and $t_k(n)$, the number of representations of n as a sum of k squares and k triangular numbers, respectively (Grosswald (1984, 1985), Moreno, Wagstaff Jr. (2006)) and also to $p_k(n)$, the number of color partitions of n (López-Bonilla et al. (2021), López-Bonilla and Morales-García (2022), López-Bonilla and Yaljá Montiel-Pérez (2021)).

We consider that

(4.1) $F(q)=\sum_{n=-\infty}^{\infty}q^{n^2}=\prod_{n=1}^{\infty}\frac{1-q^n}{1+q^n}.$

Thus due to Andrews et al. (2022), we get

(4.2) $F^k=\sum_{n=0}^{\infty}(-1)^n r_k(n)\,q^n,\quad G(q)=-2\sum_{n=1}^{\infty}n\,D(n)\,q^n,$

where $D(n)$ is the sum of the inverses of the odd divisors of n, that is, $D(n)=\sum_{odd\ d/n}\frac{1}{d}$, and $r_k(n)$ is the number of representations of a positive integer n as a sum of k squares, such that representations with different orders and signs are counted as distinct (Grosswald (1984, 1985), Lam-Estrada and López-Bonilla (2021), Moreno, Wagstaff Jr. (2006))

Therefore, we find that

(4.3) $f_k(n) = (-1)^n\, r_k(n),\;\; g_n = -2n\, D(n) = n\, g_1\, D(n),\;\;\; g_1 = -2.$

Now apply (4.3) into (3.1)-(3.9) to get the expression

(4.4) $r_k(n) = \frac{2^n}{n!}\, k^n + (-1)^n \sum_{j=1}^{n-1} a(n,j)\, k^j\,,$

such that

(4.5)

$$a(n,m) = \frac{(-2)^m}{m!\,(n-m)!} \sum_{j=1}^{n-m} \binom{m}{j} j!\; B_{n-m,j}\big(1!\,D(2), 2!\,D(3), \dots, (n-m-j+1)!\,D(n-m-j+2)\big).$$

The relations (4.4) and (4.5) generate polynomials in k for several values of n, given by

(4.6) $r_k(1) = 2k,\qquad r_k(2) = 2k(k-1),\qquad r_k(3) = \frac{4}{3}\, k(k-1)(k-2),$

$$r_k(4) = \frac{2}{3}\, k[3(2k-1) + k(k-1)(k-5)],\quad r_k(5) = \frac{4}{15}\, k(k-1)[3(2k-3) + k(k-4)(k-5)],$$

$$r_k(6) = \frac{4}{45}\, k(k-1)(k-2)\,[45 + (k-3)(k-4)(k-5)],$$

$$r_k(7) = \frac{8}{315}\, k(k-1)(k-2)(k-3)\big(k^3 - 15k^2 + 74k - 15\big),\;\dots$$

Further consider

(4.7) $F(q) = (q\,;q)_\infty = \prod_{n=1}^{\infty} (1-q^n)\,.$

Then in (4.7) apply the results due to Andrews et al. (2022) and obtain that

$$F^k = \sum_{n=0}^{\infty} p_k(n)\, q^n\,,\qquad G(q) = -\sum_{n=1}^{\infty} \sigma(n)\, q^n\,,$$

where $p_k(n)$ is the number of partitions with k colors (Jakimczuk (2022), López-Bonilla et al. (2021), López-Bonilla and Morales-García (2022), López-Bonilla and Yaljá Montiel-Pérez (2021), Pathan et al. (2022)) and $\sigma(n)$ is the sum of the divisors of n and hence to get

(4.8) $f_k(n) = p_k(n),\qquad g_n = -\sigma(n) = g_1\, \sigma(n),\quad g_1 = -1,$

and the Eqns. (3.1)-(3.9) imply the relation

(4.9) $p_k(n) = \frac{(-1)^n}{n!}\, k^n + \sum_{j=1}^{n-1} a(n,j)\, k^j\,,$

where,

(4.10)

$$a(n,m) = \frac{(-1)^m}{m!\,(n-m)!} \sum_{j=1}^{n-m} \binom{m}{j} j!\; B_{n-m,j}\left(\frac{1!}{2}\,\sigma(2), \frac{2!}{3}\,\sigma(3), \dots, \frac{(n-m-j+1)!}{n-m-j+2}\,\sigma(n-m-j+2)\right),$$

which allows generate the following polynomials

(4.11) $p_k(1) = -k\,,\qquad p_k(2) = \frac{1}{2!}\, k\,(k-3),\qquad p_k(3) = -\frac{1}{3!}\, k\,(k-1)\,(k-8),$

$$p_k(4) = \frac{1}{4!}\, k\,(k-1)\,(k-3)\,(k-14),\qquad p_k(5) = -\frac{1}{5!}\, k\,(k-3)\,(k-6)\,\big(k^2 - 21k + 8\big),\dots$$

Further consider

(4.12) $$F(q)=\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}=\prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{2n-1}}=\prod_{n=1}^{\infty}(1+q^n)^2(1-q^n).$$

Therefore by Eqn. (4.12) and due to Andrews et al. (2023), we have

$$F^k=\sum_{n=0}^{\infty} t_k(n)\,q^n, \qquad G(q)=-\sum_{n=1}^{\infty} n\,T(n)\,q^n,$$

where $t_k(n)$ is the number of representations of n as the sum of k triangular numbers, such that representations with different orders are counted as unique, and

(4.13) $$T(j)=\sum_{d|j}\frac{1+2(-1)^d}{d}=\frac{1}{j}\sum_{d|j}(-1)^d d\,.$$

Hence there exists

(4.14) $$f_k(n)=t_k(n). \qquad g_n=-n\,T(n),\quad g_1=1,$$

and from (3.1)-(3.9), we get

(4.15) $$t_k(n)=\frac{1}{n!}k^n+\sum_{j=1}^{n-1} a(n,j)\,k^j\,,$$

where

(4.16)

$$a(n,m)=\frac{1}{m!\,(n-m)!}\sum_{j=1}^{n-m}(-1)^j\binom{m}{j}j!\,B_{n-m,j}\big(1!\,T(2),2!\,T(3),\dots,(n-m-j+1)!\,T(n-m-j+2)\big),$$

generating the polynomials

(4.17)

$$t_k(1)=k,\quad t_k(2)=\frac{k}{2}(k-1),\quad t_k(3)=\frac{k}{6}\big(k^2-3k+8\big),\quad t_k(4)=\frac{k}{24}(k-1)\big(k^2-5k+30\big),\dots\ .$$

Remark 1. By making an appeal to the results of the Theorems 2.1 and 3.1, it is possible to deduce the useful formulas

(4.18) $$a(n,1)=\frac{1}{n}g_n\,,\quad n\ge 1;\qquad a(n,n-1)=\frac{1}{2\,(n-2)!}(g_1)^{n-2}g_2\,,\quad n\ge 2,$$

$$a(n,n-2)=\frac{1}{3\,(n-3)!}(g_1)^{n-3}g_3+\frac{1}{8\,(n-4)!}(g_1)^{n-4}(g_2)^2\,,\qquad n\ge 3,$$

and

$$a(n,n-3)=\frac{1}{4\,(n-4)!}(g_1)^{n-4}g_4+\frac{1}{6\,(n-5)!}(g_1)^{n-5}g_3\,g_2+\frac{1}{48\,(n-6)!}(g_1)^{n-6}(g_2)^3\,,\quad n\ge 4,\dots$$

$$a(n,n-l)=\frac{1}{n}\sum_{j=1}^{l+1} g_j\,a(n-j,\,n-l-1),\qquad 0\le l\le n-1,\quad n\ge 1\,.$$

Remark 2. With (3.1) and (3.3), it is immediate that

(4.19) $$\sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n}{k}f_k(n)=-a(n,1)+(-1)^n n!\sum_{j=1}^{n-1}a(n,j+1)\,S_j^{[n]}=-\frac{1}{n}g_n\,,$$

because $a(n,1)=\frac{1}{n}g_n$ and the Stirling numbers of the second kind verify the property $S_j^{[n]}=0,\ j<n\,.$

Remark 3. Three interesting recurrence relations are obtained as

(4.20) $$p_{k+1}(n)=\sum_{j=0}^{n} a(j)\,p_k(n-j), \qquad r_{k+1}(n)=\sum_{j=0}^{n} b(j)\,r_k(n-j),$$

$$t_{k+1}(n)=\sum_{j=0}^{n} c(j)\,t_k(n-j),$$

where

(4.21) $$a(j) = \begin{cases} 0, & j \neq \frac{m}{2}(3m+1), \\ (-1)^m, & j = \frac{m}{2}(3m+1), \quad m = 0, \ \pm 1, \ \pm 2, \dots \end{cases}$$

$$b(j) = \begin{cases} 2, & n = m^2, \ m \geq 1, \\ 1, & n = 0 \\ 0, & otherwise, \end{cases} ,$$

$$c(j) = \begin{cases} 1, & n = \frac{m}{2}(m+1), \quad m \geq 0, \\ 0, & otherwise, \end{cases}$$

which are immediate from the following result for any analytic function *F(q)*, given by

(4.22) " If $(F(q))^k = \sum_{n=0}^{\infty} f_k(n)\, q^n$ then $f_{k+1}(n) = \sum_{j=0}^{n} f_1(j)\, f_k(n-j)$ ".

In fact we have

(4.23) $$\sum_{n=0}^{\infty} f_{k+1}(n)\, q^n = F^{k+1} = F\,F^k = \left(\sum_{m=0}^{\infty} f_1(m)\, q^m \right)\left(\sum_{l=0}^{\infty} f_k(l)\, q^l \right),$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} f_1(j)\, f_k(n-j) \right),$$

in accordance with (4.22). If $F(q) = (q;q)_\infty$ then $f_k(n) = p_k(n)$ and $f_1(n) = p_1(n) = a(n)$, thus (4.22) implies the first relation in (4.20); for $F(q) = \prod_{j=1}^{\infty} \frac{1-q^j}{1+q^j}$. Further, we obtain that $f_k(n) = (-1)^n\, r_k(n)$ and $f_1(n) = (-1)^n\, b(n)$, so that (4.22) gives the second recurrence in (4.20); and if $F(q) = \prod_{j=1}^{\infty} (1+q^j)^2\,(1-q^j)$ then $f_k(n) = t_k(n)$ and $f_1(n) = c(n)$.Therefore (4.22) generates the third identity given in (4.20).

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