Hypersurface Properties of Almost Contact Kaehlerian Manifolds

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Abstract: Tashiro (1963) studied contact structure of hypersurfaces in complex manifold. After that, Okumura (1964), (1965) deliberate cosymplectic hypersurfaces in Kaehlerian manifold with constant holomorphic sectional curvature. Also, Negi and Chauhan (2021) have premeditated generating vector fields of the metric semi-symmetric connection on almost hyperbolic Kaehlerian manifolds. The aim of this paper, calculated the geometric properties and hypersurfaces of almost contact Kaehlerian manifold; several models of hyperspaces of almost contact Kaehlerian manifold and some theorems have been proved.

Key Words: Almost contact Kaehlerian manifolds, Differentiable manifolds, Complex manifolds and Hypersurfaces.
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1. Introduction

On a $(2n-1)$-dimensional real differentiable manifold M^{2n-1} with local coordinates system $\{x^i\}$, if tensor field Θ_j^i, Contravariant and Covariant vector field ξ^i and Ψ_i exist and satisfying the relations:

$$\xi^i \Psi_i = 1, \tag{1.1}$$

$$Rank\,(\Theta_j^i) = 2n-2, \tag{1.2}$$

$$\Theta_j^i \xi^i = 0, \qquad \Theta_j^i \Psi_i = 0 \tag{1.3}$$

$$\Theta_j^i \Theta_k^j = -\delta_k^i + \xi^i \Psi_k, \tag{1.4}$$

Then the set $(\Theta_j^i, \xi^i, \Psi_j)$ is known as an almost contact structure. The manifold with this structure is called an almost contact manifold. Sasaki (1960) verified that an almost contact manifold always admits a positive definite Riemannian metric tensor g_{jk} satisfying

$$g_{ji}\xi^i = \Psi_j, \tag{1.5}$$

$$g_{ji}\Theta_k^f \Theta_r^i = g_{kr} - \Psi_r \Psi_k, \tag{1.6}$$

The Riemannian metric g_{ji} with above properties is defined as an associated Riemannian metric and the almost contact manifold with this metric is called an almost contact metric manifold. If M^{2n-1} is $(2n-1)$-dimension differentiable manifold and there is defined over M^{2n-1} a differentiable form $\boldsymbol{\eta}$ having the property that

$\Psi \wedge d\Psi \wedge \ldots \wedge d\Psi$ (n-1 times) $\neq 0$ (1.7)

On M^{2n-1}, then M^{2n-1} is defined as a contact manifold.

If there is a function ρ in an almost contact manifold which takes now here zero value and satisfies a relation

$$2\rho\Theta_{ji} = \partial_j \Psi_i - \partial_i \Psi_j, \tag{1.8}$$

Then the rank of the matrix(Θ_{ji}) being $(2n-2)$, we have:

$$\Psi \wedge \Theta \ldots \wedge \Theta (n-1\ times) = \rho^{-n} d\Psi \wedge \ldots \wedge d\Psi (n-1\ times) \neq 0.$$

Which implies that the manifold is necessarily a contact manifold [Boothby and Wang (1968)]. So, we call such an almost contact manifold a contact metric manifold.

2. Hypersurfaces Of Almost Contact Kaehlerian Manifold.

Suppose M^{2n} be a differentiable manifold of $2n$-dimensional with local coordinates $\{X^k\}$. If in M^{2n} there is a tensor field F_λ^k which satisfies:

$$F_\lambda^\mu F_\mu^\kappa = -\delta_\lambda^\kappa, \tag{2.1}$$

The tensor F_λ^κ is known as an almost complex structure and M^{2n} with this structure as an almost complex manifold. In an almost complex manifold M^{2n} there always exists a Riemannian metric tensor satisfying:

$$G_{\kappa\lambda} F_\mu^\kappa F_\nu^\lambda = G_{\mu\nu}, \tag{2.2}$$

This is called a Hermitian metric. The pair $(F_\lambda^\kappa, G_{\lambda\kappa})$ together with above properties is called an almost Hermitian structure and the manifold M^{2n} an almost Hermitian manifold. In an almost Hermitian manifold, if the almost complex structure F_λ^κ satisfies the condition:

$$\nabla_\mu F_{\lambda\kappa} = 0, \tag{2.3}$$

Where ∇ denotes the covariant differentiation with respect to the Hermitian Metric, then the manifold is said to be Kaehlerian.

Let M^{2n} is an almost Hermitian manifold with local coordinates $\{X^\kappa\}$ and $(F_\lambda^\kappa, G_{\lambda\kappa})$ be the almost Hermitian structure. We denote by M^{2n-1} the differentiable hypersurfaces with parametric representation $X^\kappa = X^\kappa(x^l)$.

Let the hypersurfaces M^{2n-1} be Orientable, we put $B_i^\kappa = \partial_i X^\kappa (\partial_i = \partial/\partial x^i)$. Then 2n-1 vectors B_i^κ generate the tangent hyperplane of M^{2n-1} at every point of M^{2n-1}. Suppose C^κ be the unit normal vector to the hypersurface. The $2n$ vectors B_i^κ, C^κ being linearly independent, we regard that they form a local basis of a vector space at each point of M^{2n-1}. If we denote by (B_κ^i, C_κ) the dual basis of (B_i^κ, C^κ), it follows that:

$$B_i^\kappa B_\kappa^i = \delta_i^\kappa,\ \ B_i^\kappa C_\kappa = 0,\ \ B_\kappa^i C^\kappa = 0,\ \ C^i C_\kappa = 1, \tag{2.4}$$

$$B_i^\kappa B_\lambda^i + C^\kappa C_\lambda = \delta_\kappa^\delta. \tag{2.5}$$

The Riemannian metric g_{ji} on M^{2n-1} is give by:

$g_{ji} = G_{\lambda\kappa} B_j^\lambda B_i^\kappa$. (2.6)

This is defined as induced Riemannian metric ∇_j denotes the covariant differentiation with respect to the induced metric, so we have following Gauss and Weingarten equations for the hypersurface respectively:

$\nabla_j B_i^\kappa = H_{ji} C^\kappa$, (2.7)

$\nabla_j C_\kappa = -H_{ji} B_\kappa^i$, (2.8)

Where H_{ji} = Second fundamental tensor of the hypersurface.

Now we put:

$\Theta_j^i = B_j^\lambda F_\lambda^\kappa B_\kappa^i$, (2.9)

$\Psi_j = B_j^\mu F_\mu^\lambda C_\lambda = B_j^\mu F_{\mu\lambda} C^\lambda$, (2.10)

Then the set $(\Theta_j^i, g^{ir}, \Psi_r, \Psi_j)$ defines an almost contact structure and the induced Riemannian metric g_{ji} is the associated one.

Now, we use almost contact hypersurface for a hypersurface with the induced almost contact structures. To get the covariant derivatives of Ψ_i and Θ_{ji} in terms of the second fundamental tensor H_{ji} , we differentiate (2.9), (2.10) covariantly and we get:

$\nabla_j \Psi_i = B_i^\mu B_\nu^j \bar{\nabla}_\nu F_{\mu\lambda} C^\lambda - \Theta_i^r H_{ji}$, and $\nabla_j \Theta_{ir} = B_i^\mu B_j^\kappa \bar{\nabla}_\kappa F_{\mu\lambda} B_r^\lambda + \Psi_i H_{jr} - \Psi_r H_{ji}$.

If $\bar{M}^{2n}$ is Kaehlerian manifold, the above two equations can be written as:

$\nabla_j \Psi_i = -\Theta_i^r H_{ji}$, (2.11)

$\nabla_j \Theta_{ir} = \Psi_i H_{jr} - \Psi_r H_{ji}$. (2.12)

Now, we denote $\bar{M}^{2n}$ by Kaehlerian manifold and M^{2n-1} by its almost contact hypersurface. Let M^{2n-1} is a contact metric manifold [Kurtia (1963)] that is, M^{2n-1} satisfies the relation (1.8) and we call the hypersurface of an almost contact Kaehlerian manifold, the condition (1.8) is equivalent to:

$H_j^r \Theta_r^i + \Theta_j^r H_r^i = 2\rho \Theta_j^i$. (2.13)

Theorem (2.1): In hypersurface M^{2n-1} of almost contact Kaehlerian manifold, at each point of M^{2n-1} the vector Ψ^i is a characteristic vector of the second fundamental tensor H_j^i.

Proof: Transvecting (2.13) with Ψ^i, we get $\Theta_r^i H_j^r \Psi^j = 0$.

Operating Θ_i^r to thus obtained vector, we have

$H_{ji} \Psi^j = \alpha \Psi_i$, $(\alpha = H_{ji}\, \Psi^j\, \Psi^i)$. (2.14)

This completes the proof.

The functions ρ and α express the mean curvature of a hypersurface of almost contact Kaehlerian manifold. Transvecting (2.13) with Θ_{ik}, we obtain:

$H_{jk} + H_{ri}\, \Theta_j^r \Theta_k^i - 2\rho g_{jk} - (\alpha - 2\rho)\Psi_j \Psi_k = 0$, (2.15)

because of (2.14). Hence we have:

$H_r^r = \alpha + 2(n-1)\rho$. (2.16)

3. Several Models Of Hypersurfaces On Almost Contact Kaehlerian Manifolds.

Theorem (3.1): The hypersurface defined by (3.1) in Euclidean space E^{2n}, that is, a product manifold of n-dimensional Euclidean space with $(n-1)$-dimensional sphere is hypersurface of almost contact Kaehlerian manifold.

Proof: Let E^{2n} be a Euclidean space with rectangular coordinates $X^k(k=1,2,\dots,2n)$ and M^{2n-1} be a hypersurface defined by the equation:

$$\sum_{\alpha=1}^{n}(X^{n+\alpha})^2 = a^2. \quad (3.1)$$

This is a product manifold of $(n-1)$ dimensional sphere S^{n-1} with n-dimensional Euclidean space E^n. The unit normal vector C^k to the hypersurface has the components:

$$C^k = (0,\dots,0,a^{-1}X^{n+1},\dots,a^{-1}X^{2n}). \quad (3.2)$$

Differentiating (3.1), we have

$$\sum_{\alpha=1}^{n} X^{n+\alpha} B_j^{n+\alpha} = 0, \quad (3.3)$$

That is, from (3.2),

$$\sum_{\alpha=1}^{n} B_j^{n+\alpha} C^{n+\alpha} = 0. \quad (3.4)$$

By virtue of (3.1) and (3.2), it follows that:

$$H_{kj} = -a^{-1}\sum_{\alpha=1}^{n} B_k^{n+\alpha} B_j^{n+\alpha}. \quad (3.5)$$

So, we know that:

$$\nabla_k\Psi_j - \nabla_j\Psi_k = -H_{kr}\Theta_j^r + H_{jr}\Theta_k^r = a^{-1}\sum_{\alpha=1}^{n}(B_k^{n+\alpha}B_r^{n+\alpha}\Theta_j^r - B_j^{n+\alpha}B_r^{n+\alpha}\Theta_k^r). \quad (3.6)$$

However, in general, it follows that:

$$B_r^\lambda\Theta_j^r = B_r^\lambda B_\mu^r F_\nu^\mu B_j^\nu = (\boldsymbol{\delta}_\mu^\lambda - \boldsymbol{C}^\lambda \boldsymbol{C}_\mu)F_\nu^\mu B_j^\nu = F_\nu^\lambda B_j^r - \boldsymbol{C}^\lambda\Psi_j,$$

And consequently:

$$\sum_{\alpha=1}^{n} B_k^{n+\alpha} B_j^{n+\alpha}\Theta_j^r = \sum_{\alpha=1}^{n} B_k^{n+\alpha}(F_\mu^{n+\alpha}B_j^n - C^{n+\alpha}\Psi_j).$$

In E^{2n} the almost complex structure F_μ^λ having the following components

$$(F_\mu^\lambda) = \begin{pmatrix} 0 & \delta_\beta^\alpha \\ -\delta_\beta^\alpha & 0 \end{pmatrix} \quad (\alpha,\beta = 1,\dots,n),$$

The above equation is reduced to

$$\sum_{\alpha=1}^{n} B_k^{n+\alpha} B_r^{n+\alpha}\Theta_j^r = \sum_{\alpha,\beta=1}^{n} B_k^{n+\alpha}\delta_\beta^\alpha B_j^\beta = \sum_{\alpha=1}^{n} B_k^{n+\alpha} B_j^\alpha,$$

Because of (3.4). Substituting this into (3.6), we have

$$\partial_k\Psi_j - \partial_j\Psi_k = a^{-1}\sum_{\alpha=1}^{n}(B_k^{n+\alpha}B_j^\alpha - B_j^{n+\alpha}B_k^\alpha). \quad (3.7)$$

On the other hand, we have:

$$\Theta_{kj} = B_k^\lambda F_{\lambda\mu} B_j^\mu = \sum_{\alpha,\beta=1}^{n}\left(B_k^{n+\alpha}\delta_{\alpha\beta}B_j^\beta - B_j^{n+\beta}\delta_{\alpha\beta}B_k^\alpha\right) = \sum_{\alpha=1}^{n}(B_k^{n+\alpha}B_j^\alpha - B_k^\alpha B_j^{n+\alpha}).$$

Thus, we get:

$$\partial_k\Psi_j - \partial_j\Psi_k = a^{-1}\Theta_{kj}.$$

Hence it is proved.

Theorem (3.2): A totally umbilical hypersurface of almost contact Kaehlerian manifold with non-zero mean curvature is a hypersurface.

Proof: We consider a totally umbilical hypersurface M^{2n-1} of almost contact Kaehlerian manfold $\bar{M}^{2n}$ then it satisfies that:

$H_{ji} = \sigma g_{ji}$,

From which, $\nabla_k \Psi_j - \nabla_j \Psi_k = -\,\Theta_j^r H_{rk} + \Theta_k^r H_{rj} = 2\sigma \Theta_{kj}$.

This shows that in almost contact Kaehlerian manifold an umbilical hypersurface is a hypersurface.

Similarly, we have found the following:

Theorem (3.3): A $(2n-1)$ - dimensional sphere is a contact hypersurface of a 2n-dimensional Euclidean space.

Theorem (3.4): The hypersurface of a Fubinian manifold is called hypersurface of almost contact Kaehlerian manifold.

Proof: Let K be a complex number space of complex dimensional n and denote its coordinates by Z^{α} and their conjugates by $Z^{\bar{\alpha}}$.

For arbitrary k, we put $\Theta = \frac{\{\log(1+2k\sum_{\alpha=1}^{n} Z^{\alpha} Z^{\bar{\alpha}})\}}{2k} \equiv (logS)/2k,$

And $G_{\alpha\bar{\beta}} = \partial_{\alpha}\partial_{\beta}\Theta,\ G_{\alpha\beta} = G_{\bar{\alpha}\beta} = 0\ (\alpha, \beta = 1, \ldots, n),$

That is $G_{\alpha\bar{\beta}} = G_{\bar{\beta}\alpha} = (S\delta_{\alpha\beta} - 2kZ^{\bar{\alpha}}Z^{\beta})/S^2,\ \ G_{\alpha\beta} = G_{\bar{\alpha}\bar{\beta}} = 0.$

Then the connected component of K, where S doesn't vanish, is a Kaehlerian manifold with respect to the metric $G_{\lambda k}$. we call the Kaehlerian manifold is Fubinian manifold [Tashiro and Tachibana (1963)] and denote it by $\bar{M}$. A Fubinian manifold if the Christoffel symbols of $\bar{M}$ are:

$$\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = -2k(\delta_r^{\alpha} z^{\beta} + \delta_{\beta}^{\alpha} z^{\gamma})/S \qquad (3.8)$$

and their conjugates.

Consider in $\bar{M}$ a hypersurface M defined by:

$$\sum_{\alpha=1}^{n} z^{\alpha} Z^{\bar{\alpha}} = a^{t}. \qquad (3.9)$$

The unit normal vector C^{k} and the induced Riemannian metric on M are respectively given by:

$$C_k = (z^{\bar{\alpha}}/\sqrt{2}aS, z^{\alpha}/\sqrt{2}aS), \qquad (3.10)$$

$$g_{ji} = S^{-1}\sum_{\alpha=1}^{n}\left(B_i^{\bar{\alpha}} B_j^{\alpha} + B_i^{\alpha} B_j^{\bar{\alpha}}\right) - 4a^2 k \sum_{\alpha,\beta=1}^{n}(B_j^{\alpha} C_{\alpha} B_i^{\bar{\alpha}} C_{\alpha} + B_j^{\bar{\alpha}\alpha} C_{\alpha} B_i^{\alpha} C_{\alpha}). \qquad (3.11)$$

Since F_{μ}^{λ} has the component: $\left(F_{\mu}^{\lambda}\right) = \begin{pmatrix} \sqrt{-1}\delta_{\beta}^{\alpha} & 0 \\ 0 & -\sqrt{-1}\delta_{\beta}^{\alpha} \end{pmatrix}$

Then, from (2.10) we get:

$$\Psi_j = \sqrt{-1}\sum_{\alpha=1}^{n}\left(B_j^{\alpha} C_{\alpha} - B_j^{\alpha} C_{\bar{\alpha}}\right). \qquad (3.12)$$

On the other hand, differentiating (3.9), we get: $\sum_{\alpha=1}^{n}\left(B_i^{\alpha} z^{\bar{\alpha}} - B_j^{\bar{\alpha}} z^{\alpha}\right) = 0.$

This implies: $\sum^{n}{}_{\alpha=1}\left[H_{ji}(C^{\alpha} z^{\bar{\alpha}} + C^{\bar{\alpha}} z^{\alpha}) + B_i^{\alpha} B_j^{\bar{\alpha}} + B_i^{\bar{\alpha}} B_j^{\alpha} - B_i^{k} B_j^{\lambda}\left(\begin{Bmatrix} \alpha \\ k\lambda \end{Bmatrix} z^{\bar{\alpha}} + \begin{Bmatrix} \bar{\alpha} \\ k\lambda \end{Bmatrix} z^{\alpha}\right)\right] = 0.$

Substituting (3.8) and (3.10) into the above and making use of C^k as the unit normal vector to M, we get:

$$H_{ji} = \frac{-1}{\sqrt{2}aS}\left\{\sum_{\alpha=1}^{n}(B_i^{\alpha}B_j^{\bar{\alpha}} + B_i^{\bar{\alpha}}B_j^{\alpha}) + \sum_{\beta,\gamma=1}^{n} 8ka^2 S(B_i^{\beta}C_{\beta}B_j^{\gamma}C_{\gamma} + C_{\bar{\beta}}B_j^{\bar{\gamma}}C_{\bar{\gamma}})\right\}. \quad (3.13)$$

By means of (3.11), (3.12) and (3.13), we get:

$$H_{ji} = -a^{-1}(g_{ji} + 2a^2k\Psi_j\Psi_i)/\sqrt{2}. \quad (3.14)$$

Consequently it follows that:

$$\partial_j\Psi_i - \partial_i\Psi_j = \nabla_j\Psi_i - \nabla_i\Psi_j = -\sqrt{2}a^{-1}\theta_{ji},$$

because of (2.11). This means that the hypersurface M is a contact hypersurface.

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