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On Conformal Symmetric Tensor of Kaehlerian Manifolds

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Abstract: Chaki (1956) has obtained some theorems on recurrent and ricci-recurrent spaces. After that, Negi (2017) has calculated theorems on almost product and decomposable spaces. Also, Negi et al. (2019) has studied an analytic HP-transformation in almost Kaehlerian spaces. In this paper, we have defined and considered on symmetric tensor and conformal symmetric tensor of Kaehlerian manifolds and some theorems established.

Key Words: Kaehlerian manifolds, conformal symmetric tensor and Ricci-recurrent spaces.

MSC 2010: 53C15, 53C55, 53B35.

Introduction:

An ***n***-dimensional Kaehlerian manifolds K_n $(n \geq 2)$ will be called conformal symmetric tensor if the conformal curvature tensor will be:

$$C^h{}_{ijk} = R^h{}_{ijk} + \frac{1}{n-2}(\delta^h{}_j R_{ik} - \delta^h{}_k R_{ij} + g_{ik}R^h{}_j - g_{ij}R^h{}_k) + \frac{R}{(n-1)(n-2)}(\delta^h{}_k g_{ij} - \delta^h{}_j g_{ik}) \qquad (1.1)$$

$$C^h{}_{ijk,l} = 0 \qquad (1.2)$$

where (,) denote covariant differentiation with respect to the metric tensor g_{ij} of K_n.

By easy calculation, we have

$$C_{ijkl}\, C^{ijkl} = R_{ijkl}\, R^{ijkl} - \frac{4R_{ij}R^{ij}}{(n-2)} + \frac{2R^2}{(n-1)(n-2)} \qquad (1.3)$$

used for K_n, then $(C_{ijkl}\, C^{ijkl})_{,m} = 0$. Hence $C_{ijkl}\, C^{ijkl}$ is constant.

While the scalar $C_{ijkl}\, C^{ijkl}$ is constant for K_n, the scalars $R_{ijkl}\, R^{ijkl}$ and $R_{ij}\, R^{ij}$ are not necessarily consequently. Thus, form (1.3) it follows that the scalar curvature of K_n is of constant scalar curvature iff the first covariant derivative of the Ricci tensor is a symmetric tensor. From (1.2) we obtain [Chaki (1956)]:

$$C^h{}_{ijk,h} = \frac{n-3}{n-2}\left[R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik}\, R_{,j} - g_{ij}\, R_{,k})\right] = 0 \qquad (1.4)$$

from (1.4) and then knowing $(n > 3)$, we get:

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)} (g_{ij} R_{,k} - g_{ik} R_{,j}) \tag{1.5}$$

If R is constant, $R_{ij,k} - R_{ik,j} = 0$. That is, the tensor $R_{ij,k}$ is a symmetric tensor. Equally, if $R_{ij,k}$ be a symmetric tensor, from (1.5) we get:

$$g_{ij} R_{,k} = g_{ik} R_{,j}$$

whence $R_{,k} = 0$, so R is constant.

Again, K_n of constant scalar curvature $R_{ij,k} - R_{ik,j} = 0$. Other than for such a recurrent space $R_{ij,kl} - R_{ij,lk}$, which is not, in general zero, can be expressed in terms of the metric tensor and the second covariant derivatives of the tensor $R^i{}_j$.

The circumstances of integrability of the equations (1.2) are:

$$C^h{}_{\alpha jk} R^\alpha{}_{ilm} + C^h{}_{i\alpha k} R^\alpha{}_{jlm} + C^h{}_{ij\alpha} R^\alpha{}_{klm} - C^\alpha{}_{ijk} R^h{}_{\alpha lm} = 0 \tag{1.6}$$

We can state (1.6) when:

$$(R^h{}_{ijk,lm} - R^h{}_{ijk,ml}) + \frac{1}{n-2} [\delta^h{}_j (R_{ik,lm} - R_{ik,ml}) - \delta^h{}_k (R_{ij,lm} - R_{ij,ml}) + g_{ik}(R^h{}_{j,lm} - R^h{}_{j,ml}) - g_{ij} (R^h{}_{k,hm} - R^h{}_{k,mh})] = 0 \tag{1.7}$$

Contracting h and l we get:

$$(R^h{}_{ijk,hm} - R^h{}_{ijk,mh}) + \frac{1}{n-2} [(R_{ik,jm} - R_{ij,km}) + (R_{ij,mk} - R_{ik,mj}) + g_{ik} (R^h{}_{j,hm} - R^h{}_{j,mh}) - g_{ij} (R^h{}_{k,hm} - R^h{}_{k,mh}) = 0 \tag{1.8}$$

If a recurrent space is of constant scalar curvature then $R_{ij,k} = R_{ik,j}$ therefore,

$$R^h{}_{ijk,h} = 0 \tag{1.9}$$

Hence (1.8) reduces to:

$$R^h{}_{ijk,mh} = \frac{1}{n-2} [(R_{im,jk} - R_{im,kj}) + (g_{ij}R^h{}_{k,mh} - g_{ik}R^h{}_{j,mh})] \tag{1.10}$$

Again in a Kaehlerian manifolds K_n , then:

$$R^h{}_{ijk,hm} - R^h{}_{ijk,mh} + R^h{}_{mjk,hi} - R^h{}_{mjk,ih} - R_{im,jk} + R_{im,kj} = 0 \tag{1.11}$$

Hence in asset of (1.9), we get:

$R_{im,kj} - R_{im,jk} = R^h{}_{ijk,mh} + R^h{}_{mjk,ih}$.

Therefore from (1.10), then:

$$R_{im,jk} - R_{im,kj} = \frac{1}{n}(g_{ik}R^h{}_{j,mh} + g_{mk}R^h{}_{j,ih} - g_{ij}R^h{}_{k,mh} - g_{mj}R^h{}_{k,ih}) \quad (1.12)$$

That is $R_{im,jk} - R_{im,kj}$ for K_n of constant scalar curvature in terms of the metric tensor g_{ij} and the second covariant derivatives of the tensor $R^h{}_j$.

2. On Symmetric Tensor of Kaehlerian Manifolds:

Theorem (2.1): Every Einstein symmetric tensor of Kaehlerian manifolds K_n is symmetric in the logic of cartan the scalar curvature of which is:

$$R^2 = \frac{n(n-1)}{2}[R_{ijkl}R^{ijkl} - C_{ijkl}C^{ijkl}].$$

Proof: Let K_n be an Einstein manifold. Then we have:

$$R_{ij} = \frac{R}{n}g_{ij}.$$

since $(n > 3)$, R is constant. Therefore $R_{ij,p} = 0$.

Hence from (1.2),

$$R^h{}_{ijk,l} = 0. \quad (2.1)$$

Since R is constant and the manifold is an Einstein manifold $R_{ij}R^{ij}$ is also constant. Hence from (1.3) it follows that $R_{ijkl}R^{ijkl}$ is also a constant. This is also obvious from (2.1).

Since $R_{ij}R^{ij} = \frac{R^2}{n}$, we have from (1.3), then:

$$C_{ijkl}C^{ijkl} = R_{ijkl}R^{ijkl} - \frac{2R^2}{n(n-1)},$$

where $R^2 = \frac{n(n-1)}{2}[R_{ijkl}R^{ijkl} - C_{ijkl}C^{ijkl}]$ (2.2)

Hence we get the result.

Theorem (2.2): Every recurrent Kaehlerian manifolds K_n is conformal smooth and Ricci-recurrent provided that $R_{ijkl,a} + \lambda_a(C_{ijkl} - R_{ijkl}) = 0$, where λ_a is a non-zero vector.

Proof: Let K_n is a recurrent space individual by a non-zero vector λ_a. Then [Brinkmann (1924)],

$$C_{ijkl,m} = \lambda_m C_{ijkl}.$$

Since $C_{ijkl,m} = 0$, $\lambda_m C_{ijkl} = 0$

whence $C_{ijkl} = 0$.

After that, we assume that K_n is a Ricci- recurrent space individual by a non-zero vector λ_a, therefore:

$$C_{ijkl,a} = R_{ijkl,a} + \frac{1}{n-2}\left(g_{ik} R_{jl,a} - g_{il} R_{jk,a} + g_{jl} R_{ik,a} - g_{jk} R_{il,a} \right) + \frac{R_{,a}}{(n-1)(n-2)}\left(g_{il}\, g_{jk} - g_{ik}\, g_{jl}\right),$$

We have in this case:

$$R_{ijkl,a} + \lambda_a \left(C_{ijkl} - R_{ijkl}\right) = 0 \qquad (2.3)$$

Conversely, if (2.3) holds:

$R_{il,a} + \lambda_a (C_{il} - R_{il}) = 0$, where $R_{il,a} = \lambda_a R_{il}$. Hence proved.

Theorem (2.3): If Kaehlerian manifolds K_n be Ricci-recurrent then its vector of recurrent is a null vector and the rank of the matrix (R_{ij}) is one.

Proof: We now assume that K_n is a Ricci-recurrent space, the vector of recurrences of which is a gradient. Then, [Rahman (1970)],

$$R_{ij,kl} - R_{ij,lk} = 0$$

Again, for a non-decomposable Ricci-recurrent space, $R = 0$.

Hence, using formula (1.10), we have

$$g_{ik} R^h{}_{j,mh} + g_{mk} R^h{}_{j,ih} - g_{ij} R^h{}_{k,mh} - g_{mj} R^h{}_{k,ih} = 0 \qquad (2.4)$$

Thus, if Kaehlerian manifolds K_n be Ricci-recurrent with its vector of recurrence as a gradient, then (2.4) holds. From non-decomposable Ricci-recurrent K_n, that is:

$R_{ij,k} = R_{ik,j} \implies \lambda_k R_{ij} = \lambda_j R_{ik}$, then:

$$R_{ij} = r\, \lambda_i \lambda_j \quad (r \neq 0) \qquad (2.5)$$

where **r** is a scalar factor of proportionality.

from (2.5), we get: $g^{ij} R_{ij} = r\, g^{ij} \lambda_i \lambda_j$.

where $g^{ij} \lambda_i \lambda_j = 0$.

Hence proved.

Theorem (2.4): Every projective-symmetric Kaehlerian manifolds K_n is symmetric tensor in the logic of cartan.

Proof: Since every Cartan-symmetric space is both projective-symmetric manifold and conformally symmetric manifold, it follows from the theorem 2.1 that Cartan-symmetric manifold are the only manifold which are both projective-symmetric manifold and conformally symmetric manifold [Canfes (2006)],

We now consider a Kaehlerian manifolds K_n in which the weyl projective curvature tensor

$$W^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-1}(\delta^h{}_k R_{ij} - \delta^h{}_j R_{ik}) \tag{2.6}$$

and

$$W^h{}_{ijk,l} = 0 \tag{2.7}$$

An n-dimensional Kaehlerian manifolds K_n satisfying (2.7) has been called Projective-symmetric manifold. We know that for such a space, $R_{ij,k}$ is a symmetric tensor in case$(n \neq 2)$. In asset of (2.7), it follows from (2.1) and (2.6) that:

$$R^h{}_{ijk,l} = -\frac{1}{n-2}(\delta^h{}_j R_{ik,l} - \delta^h{}_k R_{ij,l} + g_{ik} R^h{}_{j,l} - g_{ij} R^h{}_{k,l})$$

$$= \frac{1}{n-1}(\delta^h{}_k R_{ij,l} - \delta^h{}_j R_{ik,l}). \tag{2.8}$$

Hence, we get:

$$g_{ik} R^h{}_{j,l} - g_{ij} R^h{}_{k,l} = \frac{1}{n-1}(\delta^h{}_k R_{ij,l} - \delta^h{}_j R_{ik,l}), \tag{2.9}$$

or

$$g_{ik} R_{tj,l} - g_{ij} R_{tk,l} = \frac{1}{n-1}(g_{tk} R_{ij,l} - g_{tj} R_{ik,l}) \tag{2.10}$$

Therefore $R_{tijk,l} = 0$.

Thus, symmetric tensor condition proved.

3. On Conformal Symmetric Tensor of Kaehlerian Manifolds:

Theorem (3.1): Given in terms of the conformal symmetric tensor, that the conformal map of a Kaehlerian Manifolds K_n be a recurrent space specified by a non- zero vector λ_p is that following:

$$\lambda_l C_{hijk} - 2\, C_{hijk}\sigma_l = -[C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j + C_{hijl}\sigma_k$$
$$- \sigma^s (g_{hl} C_{sijk} + g_{il} C_{hsjk} + g_{jl} C_{hisk} + g_{kl} C_{hijs})] \tag{3.1}$$

Proof: Let K_n^* (n >3) be a space with the metric tensor:

$$g^*{}_{ij} = e^{2\sigma} g_{ij} \tag{3.2}$$

Where g_{ij} is the metric tensor of a K_n.

Then K_n^* is said to be a conformal map of K_n. A conformal map shall be called proper if it arises from a function σ for which $\Delta_1\sigma \neq 0$. If $C^*{}_{ijk}$ and $C^{*t}{}_{ijk}$ be Weyl's conformal curvature tensors for K_n and K_n^*, then it is known that [Walker (1950)]:

$C^{*t}{}_{ijk} = C^t{}_{ijk}$

Hence

$$C^*{}_{hijk} = e^{2\sigma} C_{hijk} \tag{3.3}$$

Let a semi – colon denote covariant differentiation with respect to $g^*{}_{ij}$. Writing $\sigma_i = \frac{\partial\sigma}{\partial x^i}$ and $\sigma^j = g^{ij}\sigma_i$

We obtain from (3.3), then:

$C^*{}_{hijk,l} = e^{2\sigma} C_{hijk,l} + 2e^{2\sigma} C_{hijk}\sigma_l.$

But

$$C_{hijk,l} = C_{hijk,l} - 4C_{hijk}\sigma_l - (C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j + C_{hijl}\sigma_k).$$
$$+ \sigma^s(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{kl}C_{hijs}).$$

Therefore

$$\begin{aligned} C^*{}_{hijk,l} &= e^{2\sigma}C_{hijk,l} - 4e^{2\sigma}C_{hijk}\sigma_l - e^{2\sigma}[C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j \\ &\quad + C_{hijl}\sigma_k - \sigma^s(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{kl}C_{hijs})] + 2e^{2\sigma}C_{hijk}\sigma_l \\ &= e^{2\sigma}C_{hijk,l} - 2e^{2\sigma}C_{hijk}\sigma_l - e^{2\sigma}[C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j \\ &\quad + C_{hijl}\sigma_k - \sigma^s(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{kl}C_{hijs})] \end{aligned} \tag{3.4}$$

If K_n be a recurrent space which is not conformally flat, then we get:

$C_{hijk,l} = 0$ but $C_{hijk} \neq 0.$

equation (3.4) can be written as:

$$\begin{aligned} C^*{}_{hijk,l} &= -2e^{2\sigma}C_{hijk}\sigma_l - e^{2\sigma}[C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j + C_{hijl}\sigma_k \\ &\quad - \sigma^s(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{kl}C_{hijs})] \end{aligned} \tag{3.5}$$

If K_n^* be a recurrent space specified by a non- zero vector λ_p, then

$C^*{}_{hijk,l} = \lambda_l C^*{}_{hijk} = \lambda_l e^{2\sigma} C_{hijk}.$

equation (3.5) can be written as:

$$\lambda_l C_{hijk} - 2\, C_{hijk}\sigma_l = -[C_{lijk}\sigma_h + C_{hljk}\sigma_i + C_{hilk}\sigma_j + C_{hijl}\sigma_k - \sigma^s(g_{hl}C_{sijk} + g_{il}C_{hsjk} + g_{jl}C_{hisk} + g_{kl}C_{hijs})]$$

Thus, we get the result.

Theorem (3.2): A condition the conformal map of a Kaehlerian Manifolds K_n (n > 3) may be a recurrent space is that there exists a function σ is following:

$$\sigma_s C^s{}_{ijk} + \frac{1}{n-2} R_{ijk} = 0.$$

Proof: If, the conformal map of a Kaehlerian Manifolds K_n be a recurrent space then the condition reduces to (3.1) with the left-hand side equal to zero. With mentions to the equation (3.3), we have the following known equations:

$$e^{-2\sigma} R^*{}_{hijk} = R_{hijk} + g_{hk}(\sigma_{i,j} - \sigma_i\sigma_j) + g_{ij}(\sigma_{h,k} - \sigma_h\sigma_k) - g_{hj}(\sigma_{i,k} - \sigma_i\sigma_k) - g_{ik}(\sigma_{h,j} - \sigma_h\sigma_j) + (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_l\sigma, \qquad (3.6)$$

where

$$\sigma_{i,j} - \sigma_i\sigma_j = \frac{1}{n-2}(R^*{}_{ij} - R_{ij}) - \frac{1}{2(n-1)\,(n-1)}(g^*{}_{ij}R^* - g_{ij}\,R) - \frac{1}{2} g_{ij}\Delta_1\sigma \qquad (3.7)$$

The conditions of integrability of the equation (3.7) were obtained [Ficken (1939)]:

$$\sigma_s C^s{}_{ijk} = \frac{1}{n-2}(R^*{}_{ijk} - R_{ijk}) \qquad (3.8)$$

where

$$R_{ijk} = R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}(g_{ij}R_{,k} - g_{ik}R_{,j})$$

and

$$R^*{}_{ijk} = R^*{}_{ij,k} - R^*{}_{ik,j} - \frac{1}{2(n-1)}(g^*{}_{ij}R^*{}_{,k} - g^*{}_{ik}R^*{}_{,j})$$

If we now suppose that $K_n{}^*$ is a recurrent space, than $R^*{}_{ijk} = 0$.

Putting $R^*{}_{ijk} = 0$ in (3.8) we get:

$$\sigma_s C^s{}_{ijk} + \frac{1}{n-2} R_{ijk} = 0.$$

Hence we get the result.

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