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Existence of Fixed Point Under Generalized Multivalued $(\psi\text{-}F_{\mathfrak{R}})$-Contraction in Partial Metric Spaces

Smita Negi[1] • Umesh Chandra Gairola[1*]

[1]*Department of Mathematics, H.N.B. Garhwal University, BGR Campus, Pauri Garhwal-246001, Uttarakhand, India*

*Corresponding Author Email id: ucgairola@rediffmail.com

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Abstract In this paper, we introduce the notion of generalized multivalued $(\psi\text{-}F_{\mathfrak{R}})$-contraction in partial metric space endowed with an arbitrary binary relation and establish a fixed point theorem for this contraction mapping. Our result extends and generalize the result of Wardowski (Fixed Point Theory Appl. 2012:94 (2012)), Alam and Imdad (J. Fixed Point Theory Appl. 17 (4) (2015), 693–702) and Altun et al. (J. Nonlinear Convex Anal. 28 (16) (2015), 659-666). Also, we give an example to validate our result.
Mathematics Subject Classifications: 47H10, 54H25.

Keywords: Fixed Point • Binary Relation • Partial Metric Space • Generalized Multivalued $(\psi\text{-}F_{\mathfrak{R}})$-Contraction.

1. Introduction

The research on fixed point theory deals with the number of extensions and generalizations of the notion of metric space and the Banach contraction principle (Bcp). Researchers continue to search for a new way to generalize these notions in order to grow the potential of applications in different fields of mathematics as well as other sciences. Wardowski (2012) introduced the notion of F-contraction, which is one of the novel generalization of Bcp. This class of contraction has many improved versions in the literature. Altun et al. (2015) gave the multivalued version of Wardowski's result in complete metric space.

On the other side, Matthews (1994) gave the concept of partial metric space as a generalization of metric space and also proved the Bcp in such space. The specialty of such spaces is that they also include the case of non-zero self-distance and are utilized to solve the issues of computer programming. Later on, mathematicians explored partial metric spaces and found several classical fixed point results that are useful in both theory and applications (see, for example Oltra and Valero 2004; Bukatin et al. 2009; Aydi et al. 2012; Dimri and Prasad 2017).

Alam and Imdad (2015) introduced the relation-theoretic contraction principle in metric space under an arbitrary binary relation which extend the Bcp. Many researchers extend and generalize this result in different ways (see, Altun et al. 2019; Antal and Gairola 2020; Antal et al. 2021; Imdad et al. 2018 and references therein). Recently, Sawangsup et al. (2017) introduced the notion of $F_{\mathfrak{R}}$-contraction in relational metric space which extends the result of Wardowski (2012).

Our aim is to prove a fixed point theorem for generalized multivalued $(\psi\text{-}F_{\mathfrak{R}})$-contraction mapping in the context of partial metric spaces.

We include an example to demonstrate the theorem's validity and also give some remarks to show the importance of our result.

2. Preliminaries

We recall some related definitions and results needed in the sequel. Throughout the paper, we use the symbol $\mathbb{R}^+$ for $(0,\infty)$, $\mathbb{R}_0^+$ for $[0,\infty)$, $\mathbb{R}$ for $(-\infty,\infty)$, $\mathbb{N}$ for $\{1,2,3,..\}$ and $\mathbb{N}_0$ for $\{0,1,2,3,\dots\}$. Firstly, we recall the definition of F-contraction, which was introduced by Wardowski (2012).

Definition 2.1. (Wardowski 2012). Let $F:\mathbb{R}^+\to\mathbb{R}$ be a function satisfying the following conditions:
(F_1) F is strictly increasing, i.e., for all $u,v\in\mathbb{R}^+$ such that $u<v, F(u)<F(v)$;
(F_2) for each sequence $\{v_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} v_n=0$ iff $\lim_{n\to\infty} F(v_n)=-\infty$;
(F_3) there exists $\sigma\in(0,1)$ such that $\lim_{u\to 0} v^{\sigma}F(u)=0$.
The family of all functions F satisfying (F_1)-(F_3) is denoted by $\mathcal{F}$.

Definition 2.2. (Wardowski 2012). Let (Y,d) be a metric space. A mapping $S: Y\to Y$ is said to be an F-contraction on (Y,d), if there exists $F\in\mathcal{F}$ and $\gamma>0$ such that for all $u,v\in Y$,

$$d(Su,Sv)>0\Rightarrow \gamma+F(d(Su,Sv))\le F(d(u,v)).$$

In 2013, Secelean (2013) proved the following lemma:

Lemma 2.1. Let $F:\mathbb{R}\to\mathbb{R}$ be an increasing mapping and $\{v_n\}$ be a sequence of positive real numbers. Then the following assertions hold:
(i) if $\lim\limits_{n\to\infty} F(v_n)=-\infty$, then $\lim\limits_{n\to\infty} v_n=0$;
(ii) if $\inf F=-\infty$ and $\lim\limits_{n\to\infty} v_n=0$, then $\lim\limits_{n\to\infty} F(v_n)=-\infty$.

By proving Lemma 2.1, Secelean (2013) showed that the condition (F_2) in Definition 2.1 can be replaced by an equivalent following condition,
(F_2^*) $\inf F=-\infty$;
or
(F_2^{**}) there exists a sequence $\{v_n\}$ of positive real numbers such that $\lim\limits_{n\to\infty} F(v_n)=-\infty$.

Piri and Kumam (2014) generalize the result of Secelean (2013) by replacing condition (F_3) with the following:
(F_3^*) F is continuous.

Later, many authors generalize the notion of F-contraction in different ways (see, Kumari et al. 2018; Proinov 2020 and references therein). Let (F_4) $F(\inf A):=\inf F(A)$ for all $A\subset(0,\infty)$ with $\inf A>0$. We represent by $\mathcal{F}^*$ the family of all functions F that satisfy $(F_1),(F_2^*),(F_3^*)$ and (F_4).

Matthews (1994) introduced the notion of partial metric spaces and their relative topological notions as follows:

Definition 2.3. (Matthews 1994). Let Y be a non-empty set. Then a mapping $\rho: Y\times Y\to\mathbb{R}_0^+$ is said to be a partial metric on Y if for all $u,v,w\in Y$, the following conditions hold:
(ρ1) $u=v \Leftrightarrow \rho(u,u)=\rho(u,v)=\rho(v,v)$;
(ρ2) $\rho(u,u)\le\rho(u,v)$;
(ρ3) $\rho(u,v)=\rho(v,u)$;
(ρ4) $\rho(u,v)\le\rho(u,w)+\rho(w,v)-\rho(w,w)$.

The pair (Y, ρ) is called the partial metric space.

Notice that, if ρ is a partial metric on Y, then the function $\rho_d : Y \times Y \to \mathbb{R}^+$ given by

$$\rho_d(u,v) = 2\rho(u,v) - \rho(u,u) - \rho(v,v),$$

is a metric on Y. Also, each partial metric ρ on Y generates a T_0 topology τ_ρ on Y which has as a base, the family of open balls (ρ-balls) $\{B_\rho(u,\mu) : u \in Y, \mu > 0\}$, where

$B_\rho(u,\mu) = \{v \in Y : \rho(u,v) < \rho(u,u) + \mu\}$ for all $u \in Y$ and $\mu > 0$.

Definition 2.4. (Matthews 1994). If (Y, ρ) be a partial metric space. Then a sequence $\{v_n\}$ in Y is said to be

(i) convergent, if there exists $v^* \in Y$ such that $\rho(v^*, v^*) = \lim_{n\to\infty} \rho(v^*, v_n)$;

(ii) Cauchy sequence if $\lim_{m,n\to\infty} \rho(v_n, v_m)$ exists and is finite;

(iii) complete if every Cauchy sequence $\{v_n\}$ in Y converges, with respect to τ_ρ, to a point $v^* \in Y$ such that $\rho(v^*, v^*) = \lim_{m,n\to\infty} \rho(v_n, v_m)$.

Lemma 2.2. (Matthews 1994; Oltra and Valero 2004). If (Y, ρ) be a partial metric space. Then

(i) a sequence $\{v_n\}$ is a Cauchy sequence in (Y, ρ) if and only if it is a Cauchy sequence in the metric space (Y, ρ_d);

(ii) (Y, ρ) is complete if and only if the metric space (Y, ρ_d) is complete. Further,

$$\lim_{n\to\infty} \rho_d(v_n, v^*) = 0 \text{ iff } \rho(v^*, v^*) = \lim_{n\to\infty} \rho(v^*, v_n) = \lim_{m,n\to\infty} \rho(v_n, v_m). \quad (2.1)$$

Aydi et al. (2012) gave some properties of partial metric spaces for multivalued mapping by introducing the following notions:

Let (Y, ρ) be a partial metric space and $CB^\rho(Y)$ be the family of all non-empty, closed and bounded subsets of Y. For $U, V \in CB^\rho(Y)$ and $u \in Y$, the followings notations are used:

$$\rho(u, U) = \inf \{\rho(u,a), a \in U\},$$

$$\delta_\rho(U, V) = \sup \{\rho(a, V) : a \in U\},$$

$$\delta_\rho(V, U) = \sup\{\rho(b, U) : b \in V\},$$

and

$$H_\rho(U, V) = \max\{\delta_\rho(U,V), \delta_\rho(V,U)\}.$$

Proposition 2.1. (Aydi et al. 2012). Let (Y, ρ) be a partial metric space. For $U, V, W \in CB^\rho(Y)$, we have

(i) $\delta_\rho(U,U) = \sup\{\rho(a,a) : a \in U\}$;

(ii) $\delta_\rho(U,U) \leq \delta_\rho(U,V)$;

(iii) $\delta_\rho(U,V) = 0 \Rightarrow U \subseteq V$;

(iv) $\delta_\rho(U,V) \leq \delta_\rho(U,W) + \delta_\rho(W,V) - \inf_{c\in W} \rho(c,c)$.

Proposition 2.2. (Aydi et al. 2012). Let (Y, ρ) be a partial metric space. For $U, V, W \in CB^\rho(Y)$, we have

(A) $H_\rho(U,U) \leq H_\rho(U,V)$;

(B) $H_\rho(U,V) = H_\rho(V,U)$;

(C) $H_\rho(U,V) \leq H_\rho(U,W) + H_\rho(W,V) - \inf_{c\in W} \rho(c,c)$.

Lemma 2.3. (Altun et al. 2010). Let (Y, ρ) be a partial metric space and U any non-empty set in (Y, ρ), then

$$a \in \overline{U} \Leftrightarrow \rho(a, U) = \rho(a, a),$$

where $\overline{U}$ denotes the closure of U with respect to the partial metric ρ. Here notice that U is closed in (Y, ρ) if and only if $\overline{U} = U$.

Next, we recall some relevant relation-theoretic definitions.

Definition 2.5. (Lipschutz 1964). A binary relation $\mathfrak{R}$ defined on a non-empty set Y is a subset of $Y \times Y$. For $u, v \in Y$, we say that u is $\mathfrak{R}$ related to v if and only if $(u, v) \in \mathfrak{R}$.

Definition 2.6. (Alam and Imdad 2015). Let S be a self-mapping defined on a non-empty set Y. A binary relation $\mathfrak{R}$ on Y is called S-closed if for any $\mathrm{u}, \mathrm{v} \in \mathrm{S}, (u, v) \in \mathfrak{R} \Rightarrow (\mathrm{Su}, \mathrm{Sv}) \in \mathfrak{R}$.

Definition 2.7. (Alam and Imdad 2015). Let a binary relation $\mathfrak{R}$ defined on a non-empty set Y, then a sequence $\{v_n\}$ in Y is called $\mathfrak{R}$- preserving if $(v_n, v_{n+1}) \in \mathfrak{R} \; \forall \, n \in \mathbb{N}_0$.

Definition 2.8. (Alam and Imdad 2018). Let S be a self-mapping defined on a non-empty set Y. A binary relation $\mathfrak{R}$ on Y is called S-transitive if for any $u, v, w \in Y$, $(Su, Sv), (Sv, Sw) \in \mathfrak{R} \Rightarrow (Su, Sw) \in \mathfrak{R}$.

Definition 2.9. (Alam and Imdad 2017). Let (Y, d) be a metric space, $\mathfrak{R}$ a binary relation on Y and $v^* \in Y$. A mapping $S : Y \to Y$ is called $\mathfrak{R}$-continuous at v^* if for any $\mathfrak{R}$-preserving sequence $\{v_n\}$ such that $\{v_n\} \xrightarrow{d} v^*$, we have $Sv_n \xrightarrow{d} Sv^*$. S is called $\mathfrak{R}$-continuous if it is $\mathfrak{R}$-continuous at each point of Y.

Definition 2.10. (Alam and Imdad 2017). Let (Y, d) be a metric space and $\mathfrak{R}$ a binary relation on Y, then (Y, d) is called $\mathfrak{R}$-complete if every $\mathfrak{R}$-preserving Cauchy sequence in Y converges.

Let Ψ denotes the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

1. ψ is non-decreasing.
2. $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

These functions are known as c-comparison functions and for these types of functions, we have $\psi(t) < t$ for any $t > 0$.

3. Main Result

In this section, firstly inspired by (Alam and Imdad 2015; Alam and Imdad 2017; Alam and Imdad 2018; Shahzad et al. 2015), we introduce some relation-theoretic definitions in partial metric space and then we will state our main result.

Definition 3.1. Let (Y, ρ) be a partial metric space endowed with a binary relation $\mathfrak{R}$ and $S : Y \to CB^{\rho}(Y)$ be a multivalued mapping, then

(i) $\mathfrak{R}$ is called S-closed if for any $u, v \in Y$ with $(u, v) \in \mathfrak{R}$ implies that $(a, b) \in \mathfrak{R}$ for some $a \in Su$ and $b \in Sv$.

(ii) S is called $\mathfrak{R}$-continuous at $v^* \in Y$, if for any $\mathfrak{R}$- preserving sequence $\{v_n\}$ in Y with $\{v_n\} \xrightarrow{\rho} v^*$, we have $Sv_n \xrightarrow{H_\rho} Sv^*$. We say that S is $\mathfrak{R}$-continuous if it is $\mathfrak{R}$-continuous at each point of Y.

(iii) $\mathfrak{R}$ is S-transitive if for any $u, v, w \in Y$, $a \in Su$, $b \in Sv$ and $c \in Sw$, we have $(a, b) \in \mathfrak{R}, (b, c) \in \mathfrak{R}$ implies that $(a, c) \in \mathfrak{R}$.

(iv) (Y, ρ) is called $\mathfrak{R}$-complete if every $\mathfrak{R}$- preserving Cauchy sequence in Y converges.

(v) (Y,ρ) is said to be $\Re$-regular space if for all sequences $\{v_n\}$ in Y with $(v_n, v_{n+1}) \in \Re$, $\{v_n\} \xrightarrow{\rho} v^* \in Y$ implies $(v_n, v^*) \in \Re$ for all $n \in \mathbb{N}$.

Let (Y,ρ) be a partial metric space and $S: Y \to CB^{\rho}(Y)$ is a multivalued mapping, then denote by

$$Y(S;\Re) = \{u \in Y: (u,v) \in \Re \text{ for some } v \in Su\}.$$

Definition 3.2. Let (Y,ρ) be a partial metric space endowed with a binary relation $\Re$ and $S: Y \to CB^{\rho}(Y)$ be a multivalued mapping. Let

$$\kappa = \{(u,v) \in \Re : H_{\rho}(Su,Sv) > 0\}.$$

We say that S is generalized multivalued $(\psi\text{-}F_{\Re})$-contraction on Y, if there exists $F \in \mathcal{F}^*$ and $\gamma > 0$, $\psi \in \Psi$ such that for all $(u,v) \in \kappa$, the following condition hold:

$$\gamma + F\left(H_{\rho}(Su,Sv)\right) \leq F\left(\psi\left(M(u,v)\right)\right), \tag{3.1}$$

where, $M(u,v) = \max\left\{\rho(u,v), \rho(u,Su), \rho(v,Sv), \frac{\rho(u,Sv)+\rho(v,Su)}{2}\right\}$.

Definition 3.3. In a particular case of the above definition if we take $M(u,v) = \rho(u,v)$ in (3.1), then the mapping S is called a multivalued $(\psi\text{-}F_{\Re})$-contraction.

Theorem 3.1. Let (Y,ρ) be an $\Re$-complete partial metric space equipped with a binary relation $\Re$ and $S: Y \to CB^{\rho}(Y)$ be a multivalued mapping such that $\Re$ is S-transitive. Suppose the following conditions hold:

(i) $Y(S;\Re)$ is non-empty;

(ii) $\Re$ is S-closed;

(iii) S is generalized multivalued $(\psi\text{-}F_{\Re})$-contraction;

(iv) either S is $\Re$-continuous or (Y,ρ) is $\Re$-regular space.

Then S has a fixed point.

Proof. As $Y(S;\Re)$ is non-empty, then there exists $v_0 \in Y(S;\Re)$ such that $v_1 \in Sv_0$ which implies that $(v_0, v_1) \in \Re$. We construct a sequence $\{v_n\}$ in Y such that $v_n \in Sv_{n-1}$ for all $n \in \mathbb{N}$. If $v_n = v_{n+1}$ for some $n \in \mathbb{N}_0$ then v_n becomes a fixed point of S and the proof is over. Therefore, we assume that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N}_0$. As Sv_n is a closed set then by Lemma 2.3, we get $\rho(v_n, Sv_n) > 0$ and we have

$$0 < \rho(v_n, Sv_n) \leq H_{\rho}(Sv_{n-1}, Sv_n) \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Since, $(v_0, v_1) \in \Re$ then using S-closeness of $\Re$ we get $(v_1, v_2) \in \Re$. By continuing similar process, we obtain

$$(v_n, v_{n+1}) \in \Re \text{ for all } n \in \mathbb{N}_0. \tag{3.3}$$

From inequality (3.2) and (3.3), we have $(v_{n-1}, v_n) \in \kappa$ for all $n \in \mathbb{N}$. By putting $u = v_{n-1}$ and $v = v_n$ in (3.1), we have

$$\gamma + F\left(H_{\rho}(Sv_{n-1}, Sv_n)\right) \leq F\left(\psi\left(M(v_{n-1}, v_n)\right)\right). \tag{3.4}$$

Using (F_1) in (3.2), (3.4) implies

$$F\left(\rho(v_n, Sv_n)\right) \leq F\left(H_{\rho}(Sv_{n-1}, Sv_n)\right) \leq F\left(\psi\left(M(v_{n-1}, v_n)\right)\right) - \gamma. \tag{3.5}$$

The axiom (F_4) implies that there exists $v_{n+1} \in Sv_n$ such that

$$\rho(v_n, Sv_n) = \rho(v_n, v_{n+1}).$$

From (3.5), we have

$$F\big(\rho(v_n, v_{n+1})\big) \leq F\left(H_\rho(Sv_{n-1}, Sv_n)\right) \leq F\left(\psi\big(M(v_{n-1}, v_n)\big)\right) - \gamma, \tag{3.6}$$

where,

$$\begin{aligned} M(v_{n-1}, v_n) &= \max\left\{\rho(v_{n-1}, v_n), \rho(v_{n-1}, Sv_{n-1}), \rho(v_n, Sv_n), \frac{\rho(v_{n-1}, Sv_n) + \rho(v_n, Sv_{n-1})}{2}\right\} \\ &\leq \max\left\{\rho(v_{n-1}, v_n), \rho(v_{n-1}, v_n), \rho(v_n, v_{n+1}), \frac{\rho(v_{n-1}, v_{n+1}) + \rho(v_n, v_n)}{2}\right\} \\ &\leq \max\{\rho(v_{n-1}, v_n), \rho(v_n, v_{n+1})\}. \end{aligned}$$

If we suppose that $\max\{\rho(v_{n-1}, v_n), \rho(v_n, v_{n+1})\} = \rho(v_n, v_{n+1})$, then (3.6) implies that

$$F\big(\rho(v_n, v_{n+1})\big) \leq F\left(H_\rho(Sv_{n-1}, Sv_n)\right) \leq F\left(\psi\big(\rho(v_n, v_{n+1})\big)\right) - \gamma < F\big(\rho(v_n, v_{n+1})\big) - \gamma,$$

which leads to a contradiction with respect to the definition of ψ and (F_1).
So, $\max\{\rho(v_{n-1}, v_n), \rho(v_n, v_{n+1})\} = \rho(v_{n-1}, v_n)$, then from (3.6), we get

$$F\big(\rho(v_n, v_{n+1})\big) \leq F\left(H_\rho(Sv_{n-1}, Sv_n)\right) \leq F\left(\psi\big(\rho(v_{n-1}, v_n)\big)\right) - \gamma < F\big(\rho(v_{n-1}, v_n)\big) - \gamma.$$

Similarly, by induction on n, we obtain

$$F\big(\rho(v_n, v_{n+1})\big) < F\big(\rho(v_{n-1}, v_n)\big) - \gamma < \ldots < F\big(\rho(v_0, v_1)\big) - n\gamma.$$

Letting $n \to \infty$ in above expression, we get

$$\lim_{n\to\infty} F\big(\rho(v_n, v_{n+1})\big) = -\infty.$$

From (F_2^*) and Lemma 2.1, we have

$$\lim_{n\to\infty} \rho(v_n, v_{n+1}) = 0. \tag{3.7}$$

Now, we claim that $\{v_n\}$ is a Cauchy sequence, arguing by contradiction, suppose it is not a Cauchy sequence then there exists $\epsilon > 0$ and sequences of natural numbers $\{l(n)\}$ and $\{q(n)\}$ such that for $l(n) > q(n) > n$, we have

$$\rho\big(v_{l(n)}, v_{q(n)}\big) \geq \epsilon,\ \rho\big(v_{l(n)-1}, v_{q(n)}\big) < \epsilon\ \forall\, n \in \mathbb{N}. \tag{3.8}$$

Now, by triangle inequality, we have

$$\epsilon \leq \rho\big(v_{l(n)}, v_{q(n)}\big) \leq \rho\big(v_{l(n)}, v_{l(n)-1}\big) + \rho\big(v_{l(n)-1}, v_{q(n)}\big).$$

Taking limit as $n \to \infty$ and using (3.7) and (3.8), we get $\lim_{n\to\infty} \rho\big(v_{l(n)}, v_{q(n)}\big) = \epsilon$.

Next, we claim that

$$\rho\big(v_{l(n)+1}, v_{q(n)+1}\big) > 0\ \forall\, n \geq \mathbb{N}. \tag{3.9}$$

Arguing by contradiction, there exists $m \geq \mathbb{N}$, such that

$$\rho(v_{l(m)+1}, v_{q(m)+1}) = 0. \tag{3.10}$$

Using triangle inequality, we have

$$\begin{aligned} \epsilon \leq \rho\big(v_{l(m)}, v_{q(m)}\big) &\leq \rho\big(v_{l(m)}, v_{l(m)+1}\big) + \rho\big(v_{l(m)+1}, v_{q(m)}\big) \\ &\leq \rho(v_{l(m)}, v_{l(m)+1}) + \rho(v_{l(m)+1}, v_{q(m)+1}) + \rho(v_{q(m)+1}, v_{q(m)}). \end{aligned}$$

If we take limit as $m \to \infty$ in above expression then from (3.7) and (3.10), we get a contradiction. Hence inequality (3.9) will hold.

Now, we have

$$\rho(v_{l(n)+1}, Sv_{q(n)}) \leq H_\rho(Sv_{l(n)}, Sv_{q(n)}). \tag{3.11}$$

Axiom (F_4) implies that there exists $v_{q(n)+1} \in Sv_{q(n)}$ such that

$$\rho(v_{l(n)+1}, Sv_{q(n)}) = \rho(v_{l(n)+1}, v_{q(n)+1}).$$

Thus, from (3.9) and (3.11), we have

$$0 < \rho(v_{l(n)+1}, v_{q(n)+1}) \leq H_\rho(Sv_{l(n)}, Sv_{q(n)}). \tag{3.12}$$

Since $\{v_n\}$ is $\Re$-preserving sequence then by S-transitivity of $\Re$ we have $(v_{l(n)}, v_{q(n)}) \in \Re$ and therefore from (3.12), we have $(v_{l(n)}, v_{q(n)}) \in \kappa$.

Putting $u = v_{l(n)}$ and $v = v_{q(n)}$ in (3.1), we have

$$F\left(H_\rho\left(Sv_{l(n)}, Sv_{q(n)}\right)\right) \leq F\left(\psi\left(M\left(v_{l(n)}, v_{q(n)}\right)\right)\right) - \gamma. \tag{3.13}$$

Using (F_1) in (3.12), we get

$$F\left(\rho\left(v_{l(n)+1}, v_{q(n)+1}\right)\right) \leq F\left(H_\rho\left(Sv_{l(n)}, Sv_{q(n)}\right)\right)$$

which from (3.13) further implies that

$$F\left(\rho\left(v_{l(n)+1}, v_{q(n)+1}\right)\right) \leq F\left(H_\rho\left(Sv_{l(n)}, Sv_{q(n)}\right)\right) \leq F\left(\psi\left(M\left(v_{l(n)}, v_{q(n)}\right)\right)\right) - \gamma$$
$$< F\left(M\left(v_{l(n)}, v_{q(n)}\right)\right) - \gamma \tag{3.14}$$

where,

$$M\left(v_{l(n)}, v_{q(n)}\right) = \max\left\{\begin{matrix}\rho\left(v_{l(n)}, v_{q(n)}\right), \rho\left(v_{l(n)}, Sv_{l(n)}\right), \rho\left(v_{q(n)}, Sv_{q(n)}\right), \\ \dfrac{\rho\left(v_{l(n)}, Sv_{q(n)}\right) + \rho\left(v_{q(n)}, Sv_{l(n)}\right)}{2}\end{matrix}\right\}$$

$$= \max\left\{\begin{matrix}\rho\left(v_{l(n)}, v_{q(n)}\right), \rho\left(v_{l(n)}, v_{l(n)+1}\right), \rho\left(v_{q(n)}, v_{q(n)+1}\right), \\ \dfrac{\rho\left(v_{l(n)}, v_{q(n)+1}\right) + \rho\left(v_{q(n)}, v_{l(n)+1}\right)}{2}\end{matrix}\right\}.$$

Since F is continuous then taking limit as $n \to \infty$ in (3.14), we get

$$F(\epsilon) < F(\epsilon) - \gamma,$$

which gives a contradiction. So, our assumption was wrong. This shows that $\{v_n\}$ is a Cauchy sequence with respect to ρ and hence by Lemma 2.2, $\{v_n\}$ is a Cauchy sequence in (Y, ρ_d). As (Y, ρ) is $\Re$-complete then so is (Y, ρ_d). Then there exists a sequence $\{v_n\}$ which converge to some $v^* \in Y$ with respect to metric ρ_d i.e., $\rho_d(v_n, v^*) = 0$. By using Lemma 2.2, we get

$$\lim_{n\to\infty} \rho(v_n, v^*) = \rho(v^*, v^*) = \lim_{m,n\to\infty} \rho(v_n, v_m).$$

Since, $\lim_{n\to\infty} \rho(v_n, v_m) = 0$, therefore

$$\lim_{n\to\infty} \rho(v_n, v^*) = \rho(v^*, v^*) = 0. \tag{3.15}$$

Now from hypotheses (iv), if S is $\Re$-continuous then $\lim_{n\to\infty} H_\rho(Sv_n, Sv^*) = 0$.

By triangle inequality, we have

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$$\rho(v^*,Sv^*) \leq \rho(v^*,v_{n+1}) + \rho(v_{n+1},Sv^*) \leq \rho(v^*,v_{n+1}) + H_\rho(Sv_n,Sv^*).$$

By taking limit as $n \to \infty$, we get $\rho(v^*,Sv^*) = 0$. Thus, we obtain $\rho(v^*,Sv^*) = 0 = \rho(v^*,v^*)$ and from Lemma 2.3, we get $v^* \in \overline{Sv^*} = Sv^*$.

Now, consider that (Y,ρ) is $\mathfrak{R}$-regular space then we have $(v_n,v^*) \in \mathfrak{R}$ for all $n \in \mathbb{N}$. Suppose $H_\rho(Sv_n,Sv^*) > 0$, then from (3.1) we have

$$\gamma + F\left(H_\rho(Sv_n,Sv^*)\right) \leq F\left(\psi\left(M(v_n,v^*)\right)\right), \tag{3.16}$$

where,

$$M(v_n,v^*) = \max\left\{\rho(v_n,v^*), \rho(v_n,Sv_n), \rho(v^*,Sv^*), \frac{\rho(v_n,Sv^*) + \rho(v^*,Sv_n)}{2}\right\}$$
$$\leq \max\left\{\rho(v_n,v^*), \rho(v_n,v_{n+1}), \rho(v^*,Sv^*), \frac{\rho(v_n,Sv^*) + \rho(v^*,v_{n+1})}{2}\right\}.$$

We know that

$$\rho(v_{n+1},Sv^*) \leq H_\rho(Sv_n,Sv^*). \tag{3.17}$$

Using property (F_1) in (3.17), equation (3.16) implies that

$$\gamma + F(\rho(v_{n+1},Sv^*)) \leq \gamma + F\left(H_\rho(Sv_n,Sv^*)\right) \leq F\left(\psi\left(M(v_n,v^*)\right)\right) < F\left(M(v_n,v^*)\right). \tag{3.18}$$

Since F is continuous, letting $n \to \infty$ in (3.18), we get

$$\gamma + F(\rho(v^*,Sv^*)) < F(\rho(v^*,Sv^*)),$$

which is a contradiction. Hence, we have $H_\rho(Sv_n,Sv^*) = 0$.

Therefore by taking limit in (3.17) we get

$$\lim_{n\to\infty} \rho(v_{n+1},Sv^*) = \rho(v^*,Sv^*) = 0.$$

Thus, we obtain $\rho(v^*,Sv^*) = 0 = \rho(v^*,v^*)$ and from Lemma 2.3, we get $v^* \in \overline{Sv^*} = Sv^*$.

An immediate corollary of Theorem 3.1 is the following:

Corollary 3.1. Let (Y,ρ) be an $\mathfrak{R}$-complete partial metric space equipped with a binary relation $\mathfrak{R}$ and $S : Y \to CB^\rho(Y)$ be a multivalued mapping such that $\mathfrak{R}$ is S-transitive. Suppose the following conditions hold:

(i) $Y(S;\mathfrak{R})$ is non-empty;

(ii) $\mathfrak{R}$ is S-closed;

(iii) S is multivalued $(\psi\text{-}F_{\mathfrak{R}})$- contraction;

(iv) either S is $\mathfrak{R}$-continuous or (Y,ρ) is $\mathfrak{R}$-regular space.

Then S has a fixed point.

Now we validate our main result with the following example.

Example 3.1. Let $Y = \{1,2,3,4\}$ and a partial metric is defined on Y with the values given as $\rho(1,1) = 0$, $\rho(2,2) = 0.5$, $\rho(3,3) = 1$, $\rho(4,4) = 0$, $\rho(1,2) = \rho(2,1) = 1.7$, $\rho(1,3) = \rho(3,1) = 2.5$, $\rho(1,4) = \rho(4,1) = 2$, $\rho(2,3) = \rho(3,2) = 1.5$, $\rho(2,4) = \rho(4,2) = 2.3$, $\rho(3,4) = \rho(4,3) = 2$.

We define a binary relation on Y by

$$\mathfrak{R} = \{(1,1),(1,2),(1,3),(1,4),(4,1),(4,4),(2,4)\}.$$

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A multivalued mapping $S: Y \to CB^{\rho}(Y)$ is defined as

$$Su = \begin{cases} \{1,4\}, & u = 1,4, \\ \{4\}, & u = 2,3. \end{cases}$$

Clearly, (Y,ρ) is an $\Re$-complete partial metric space. Also, for $1 \in Y$ there exists $4 \in S1$ such that $(1,4) \in \Re$, so $Y(S;\Re)$ is non-empty. It is easy to see that $\Re$ is S-closed and S-transitive. Next, if $\{v_n\}$ is an $\Re$-preserving sequence such that

$$\{v_n\} \xrightarrow{\rho} v^* \text{ and } (v_n, v_{n+1}) \in \Re \text{ for all } n \in \mathbb{N}_0,$$

then $(v_n, v_{n+1}) \in \{(1,1),(1,4),(4,1),(4,4)\}$ which give rise to $\{v_n\} \subset \{1,4\}$. As $\{1,4\}$ is closed, we have $(v_n, v^*) \in \Re$. Thus, (Y,ρ) is $\Re$-regular space. Note that for $u,v \in Y$ such that $(u,v) \in \Re$ and $H_\rho(Su,Sv) > 0$, we have $(u,v) \in \{(1,2),(1,3),(2,4)\}$. Now we show that S is generalized multivalued $(\psi$-$F_{\Re})$- contraction. If we define $F(v) = \log v$ and $\psi \in \Psi$ by $\psi(u) = \frac{9u}{10}$ then we have the following cases:

(1) If $u = 1, v = 2$, then

$$\gamma + F\left(H_\rho(S1,S2)\right) = \gamma + F\left(H_\rho(\{1,4\},\{4\})\right) = \gamma + F(2) < F\left(\psi\left(M(1,2)\right)\right) = F\left(\psi(2.3)\right).$$

(2) If $u = 1, v = 3$, then

$$\gamma + F\left(H_\rho(S1,S3)\right) = \gamma + F\left(H_\rho(\{1,4\},\{4\})\right) = \gamma + F(2) < F\left(\psi\left(M(1,3)\right)\right) = F\left(\psi(2.5)\right).$$

(3) If $u = 2, v = 4$, then

$$\gamma + F\left(H_\rho(S2,S4)\right) = \gamma + F\left(H_\rho(\{4\},\{1,4\})\right) = \gamma + F(2) < F\left(\psi\left(M(2,4)\right)\right) = F\left(\psi(2.3)\right).$$

We observe that the contraction condition (3.1) is satisfied in all possible cases. Hence, all the hypotheses of Theorem 3.1 are satisfied for $\gamma = 0.03$ and therefore S has a fixed point. In this case, $u = 1,4$ are two fixed points.

On the other hand, the metric ρ_d induced by partial metric ρ is given by

$\rho_d(1,1) = \rho_d(2,2) = \rho_d(3,3) = \rho_d(4,4) = 0,\ \rho_d(1,2) = \rho_d(2,1) = 2.9,\ \rho_d(1,3) = \rho_d(3,1) = 4,\ \rho_d(1,4) = \rho_d(4,1) = 4,\ \rho_d(2,3) = \rho_d(3,2) = 1.5,\ \rho_d(2,4) = \rho_d(4,2) = 4.1,\ \rho(3,4) = = \rho_d(4,3) = 3.$

Notice that Theorem 3.1 is not applicable for the Hausdorff metric H_{ρ_d}. For $u = 1, v = 3$, we have

$$H_{\rho_d}(S1,S3) = H_{\rho_d}(\{1,4\},\{4\}) = \max\left\{\delta_{\rho_d}(\{1,4\},\{4\}), \delta_{\rho_d}(\{4\},\{1,4\})\right\} = \max\{2,4\} = 2$$

and

$$M_d(1,3) = \max\left\{\rho_d(1,3), \rho_d(1,S1), \rho_d(3,S3), \frac{\rho_d(1,S3) + \rho_d(3,S1)}{2}\right\} = \max\left\{4,0,3,\frac{7}{2}\right\} = 4,$$

such that

$$\gamma + F\left(H_{\rho_d}(S1,S3)\right) = \gamma + F(4) \nleq F\left(\psi\left(M_d(1,3)\right)\right) = F\left(\psi(4)\right).$$

Remark 3.1. Notice that in Example 3.1, for $u = 1, v = 2$, the contraction condition of Definition 3.3 is not satisfied i.e.,

$$\gamma + F\left(H_\rho(S1,S2)\right) = \gamma + F\left(H_\rho(\{1,4\},\{4\})\right) = \gamma + F(2) \nleq F\left(\psi\left(\rho(1,2)\right)\right) = F\left(\psi(1.7)\right).$$

Remark 3.2. Observe that the result of Altun et al. (2015) is not valid in Example 3.1. For $u = 4$, and $u = 3$, we have the following:

$$\gamma + F\left(H_\rho(S4, S3)\right) = \gamma + F\left(H_\rho(\{1,4\},\{4\})\right) = \gamma + F(2) \nleq F\left(M(4,3)\right) = F(2).$$

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