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TOTALLY REAL SUBMANIFOLDS OF SASAKIAN MANIFOLD WITH VANISHING CONTACT BOCHNER CURVATURE TENSOR

K.C. PETWL AND J.S. RAWAT

Department of Mathematics, H.N.B Garhwal University Campus, Badshahi Thaul, Tehri Garhwal - 249199

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ABSTRACT

By using a complex co-ordinate system Yano & Bochner indroduced an analogue of the weyl conformal curvature tensor in a Reimannian manifold Teachibaba gave a tensor expression for such curvature tensor in a real co-ordinate system. Further, A. Carrizo, Y.H.Kim and D.W.Yoon (2004) discussed "Some inequalities on totally real sumanifolds in locally conformal Kaehler space forms". In the present paper we pursue the definition and fundamental properties of a Sasakian manifold. Also, we define the curvature tensor in a Sasakian manifold and discuss in the totally real (or anti-invariant) Submanifolds of Sasakian manifolds with vanishing contact Bochner curvature tensor and some theorems will also be investigated.

Key words: Totally Real, Sasakian manifolds, submanifold, bochner curvature tensor AMS (MOS), Subject classification (2000) : 53 C25, 53 B25,53A30

INTRODUTION

Let M^{2m} be real 2m-didamensional Kaehleran manifold with the almost complex structure F, and M^n and n-dimensional Riemannian manifold isometrically immersed in M^2m. If $T_x(M^n) \perp FT_x(M^n)$ where $T_x(M^n)$ denotes the tangent spece to M^n at a point x of M^n and is identified with its image under the differential of the immersion, then we call M^n a totally real or anti-invariant submanifold of M^{2n}. Since the rank of F is 2m, we have $n \leq 2m-n$ that is $n \leq 2m$.

The totally real submanifolds of a Koehlerian manifold have been studied by chess hoah Kon, ludden, Oguie, Okumura, D.E. Blair, K.Yano. Some definitions have been used as under:

Definition-1: Let M^{2m}, $ne \geq 4$, be a Kaehlerian manifold with vanishing Bochner curvature tensor, and M^m a totally geodesic, totally real submanifold of M^{2m}. Then it is conformally flat. (D.E. Blair).

Definition-2: Let M^{2m} $n \geq 4$, be a totally umbilical, totally real submanifolds of kaehlerian manifold M^{2m} with vanishing Bochber curvature tensor, then M^m is conformally flat. (K.Yano).

Definition-3: Let m^3 be a totally geodesic, totally real submanifold of a Kaehlerian manifold M^{2m} with vanishin Bochner curvature tensor, then M^3 is conformally flat. (K.Yano)

Definition-4: Let M^{2m}, $n \geq 4$, be a totally real submanfold of an Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. If the second fundamental tensors of M^m commute. Then M^m is conformally flat. (K.Yano & S.Bochner).

1. Sasakian Manifolds

Let M^{2m+1} be a (2m+1)-dimensional differentiable manifold of calass C^{∞} covered by a system of coordinate neighborhoods $\{U; x^i\}$ in wich there are given a tensor field F_j^i of type (1,1), a vector field V^i and a 1-form T_j satisfying

(1.1) $F_j^i F_k^j = -\delta_k^i + T_k V^i, \quad F_j^i V^j = 0, \quad T_j F^j = 1,$

where and in the sequel the indices i,j,k,........... run over the range {1,2,3,.......,2m+1}. Such a set (F,V,T) consisting of a tensor field F, a vector field V and and a 1-form T is called an almost contact manifold.If the Nijenhuis tensor

(1.2) $N_{jk}^i = F_k^x \partial_x F_j^i - F_j^x \partial_x F_k^i - (\partial_k F_j^x - \partial_j F_k^x) F_x^i$

formed with F_j^i satisfies

(1.3) $N_{jk}^i + (\partial_j T_k) V^i = 0$

where $\partial_k = \frac{\partial}{\partial x^k}$, then the almost contact structure is said to be normal and the manifold is called a normal almost contact manifold.

Suppose that in a almost contact manifold there is given a Riemannian metric g_{jk} such that

(1.4) $g_{\gamma\beta} F_k^\gamma F_j^\beta = g_{kj} - T_k T_j \quad , \quad T_j = g_{ji} V^i$,

then the almost contact structure is said to be metric, and the manifold is called an almost contact metric manifold. In view of the second equation of (1.4) we shall write V_j instead of T_j in the sequel. In an almost contact metric manifold, the tensor field $F_{kj} = F_k^\alpha g_{\alpha j}$ is skew-symmetric.

If an almost contact metric structure satisfies

(1.5) $F_{kj} = \frac{1}{2} (\partial_k V_j - \partial_j V_k)$,

then the almost contact metric structure is called a contact structure. A manifold with a normal contact structure is called a Sasakian manifold.

It is well known that in a Sasakian manifold we have

(1.6) $\nabla_j V^i = F_j^i$,

(1.7) $\nabla_k F_j^i = -g_{kj} V^i + \delta_k^i V_j$,

where ∇_j denotes the operator of covariant differentiation with respect to g_{kj}.(1.6) written as $\nabla_j V_i = F_{ji}$ shows that V^i is a Killing vector field.

Equations (1.6), (1.7) and the Ricci identity

$\nabla_v \nabla_k V^i - \nabla_k \nabla_v V^i = C_{vkj}^i V^j$,

where C_{vkj}^i is the curvature tensor, given as

(1.8) $C_{vkj}^i V^j = \delta_v^i V_k - \delta_k^i V_v$

or

(1.9) $C_{vkj}^i V_i = V_v g_{kj} - V_k g_{vj}$

that is,

(4.10) $M_{iy} = -L_{ip} A_y^p - L_{ix} A_y^x$.

Thus (4.1), (4.2) and (4.3) can be written respectively as

(4.11) $C_{iqph} + (g_{ih} - V_i V_h) L_{qp} - (g_{qh} - V_q V_h) L_{ip} + L_{ih}(g_{qp} - V_q V_p) - L_{qh}(g_{ip} - V_i V_p) - (H_{ihx} H^x_{qp} - H_{qhx} H^x_{ip}) = 0$

(4.12) $(V_i L_{qp} - V_q L_{ip}) V_y - L_{iy}(g_{qp} - V_q V_p) + L_{qy}(g_{ip} - V_i V_p) + A_{iy} M_{qp} - A_{qy} M_{ip} - 2 M_{iq} A_{py} - (\nabla_i H_{qpy} - \nabla_q H_{ipy}) = 0$

(4.13) $C_{iqyx} - (V_i L_{qy} - V_q L_{iy}) V_x - (L_{ix} V_q - L_{qx} V_i) V_y + M_{iy} A_{qx} - M_{qy} A_{ix} + A_{iy} M_{qx} - A_{qy} M_{ix} - 2 M_{iq} A_{yx} + (A_{ix} A_{qy} - A_{qx} A_{iy}) + (H^t_{iy} H_{qtx} - H^t_{qy} H_{itx}) = 0$

Case I : When V^l is tangent to M^n : We now assume that n=m+1. Then the vector field $V^l$is tangent to M^n and $A_y^x = 0$. Thus (4.13) becomes

$$C_{iqyx} - A_{ix} M_{qy} + A_{qx} M_{iy} - M_{ix} A_{qy} + M_{qx} A_{iy} + (A_{ix} A_{qy} - A_{qx} A_{iy}) + (H^t_{iy} H_{qtx} - H^t_{qy} H_{itx}) = 0,$$

from which, by transvecting with $A_p^y A_h^x$ and using $A_{ix} A_p^x = g_{qp} - V_q V_p$ derived from (3.15)(i), we have

(4.14) $C_{iqyx} A_p^y A_h^x - (g_{ih} - V_i V_h) M_{qy} A_p^y + (g_{qh} - V_q V_h) M_{iy} A_p^y - M_{ix} A_h^x(g_{qp} - V_q V_p) + M_{qx} A_h^x(g_{ip} - V_i V_p) + (g_{ih} - V_i V_h)(g_{qp} - V_q V_p) - (g_{qh} - V_q V_h)(g_{ip} - V_i V_p) + (H^t_{iy} H_{qtx} - H^t_{qy} H_{itx}) A_p^y A_h^x = 0$

We now assume that the second fundamental tensors are commutative. Then from (3.19) and (4.14) we have

(4.15) $C_{iqph} + (g_{ih} - V_i V_h) N_{qp} - (g_{qh} - V_q V_h) N_{ip} + N_{ih}(g_{qp} - V_q V_p) - N_{qh}(g_{ip} - V_i V_p) + (g_{ih} - V_i V_h)(g_{qp} - V_q V_p) - (g_{qh} - V_q V_h)(g_{ip} - V_i V_p)$

where $N_{qp} = -M_{qy} A_p^y$

Now since the vector field V^h is parallel, the Riemannian manifold M^n is locally a product of M^{n-1} in M^1 generated by V^h and M^{n-1} is totally geodesic in M^n. We represent M^{n-1} in M^n by parametric equations $y^h(z^a)$ $(a, b, c, d, \ldots\ldots\ldots = 1, 2, 3, 4 \ldots\ldots (n-1))$, and put $B_b^h = \frac{\partial y^h}{\partial z^b}$. Then we have $A_P B_b^p = 0$, and the curvature tensor C_{dcba} of M^{n-1} is given by

(4.16) $C_{dcba} = C_{iqph} B^{iqph}_{dcba}$

where $B^{iqph}_{dcba} = B_d^i B_c^q B_b^p B_a^h$. Thus transvecting (4.15) with B^{iqph}_{dcba}, we have

(4.17) $C_{dcba} + g_{da} K_{cb} - g_{ca} K_{db} + K_{da} g_{cb} - K_{ca} g_{db} = 0$,

where $g_{cb} = g_{qp} B_c^q B_b^p$ is the metric tensor of $M^{n-1}$and

$K_{cb} = N_{qp} B_c^q B_b^p + \frac{1}{2} g_{cb}$.

Equation (4.17) shows that the Weyl conformal curvature tensor of M^{n-1} vanishes ,and M^{n-1} is conformally flat if n-1≥4. Thus we have

Theorem 4.1: Let M^n , $n \geq 5$, be an totally real submanifold of an Sasakian manifold M^{2n-1} with vanishing contact Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is locally a product of an conformlly flat Riemannian space and a 1-dimensional space.

Case.II : When the vector field V^i is normal to $-\frac{1}{2} H_x H^x g_{qp}$: We now consider the case in which the vector field V^i is normal to the totally real sub-manifold M^n , so that $V^i = 0$ Then from (4.11) we obtain

(4.18) $$C_{iqph} + g_{ih} L_{qp} - g_{qh} L_{ip} + L_{ih} g_{qp} - L_{qh} g_{ip} - (H_{ihx} H^x_{qp} - H_{qhx} H^x_{ip}) = 0 .$$

If M^n is umbilical, that is, if $H_{qhx} = g_{qp} H_x$, then we can write (4.18) in the form

(4.19) $$C_{iqph} + g_{ih} (L_{qp} - \tfrac{1}{2} H_X H^x g_{qp}) - g_{qh}(L_{ip} - \tfrac{1}{2} H_x H^x g_{ip}) + (L_{ih} - \tfrac{1}{2} H_x H^x g_{ih}) g_{qp} - (L_{qh} - \tfrac{1}{2} H_x H^x g_{qh}) g_{ip} = 0$$

which shows that the Weyl conformal curvature tensor of M^n vanishes . Thus we have

Theorem 4.2 : Let M^n, n≥4 be a totally umbilical totally real submanifold normal to the structure vector field V^i of a Sasakian manifold M^{2m+1} with vanishing contact Bochner curvature tensor. Then M^n is conformally flat.

Next from (4.13), we obtain

(4.20) $$C_{iqyx} + M_{iy} A_{qx} - M_{qy} A_{ix} + A_{iy} M_{qx} - A_{qy} M_{ix} + 2 M_{iq} A_{yx} + (A_{ix} A_{qy} - A_{qx} A_{iy}) + (H^t_{iy} H_{qtx} - H^t_{qy} H_{itx}) = 0$$

If n=m, which implies that $A^x_y = 0$, and the second fundamental tensors of M^n commute, then from we have

(4.21) $$C_{iqyx} - A_{ix} M_{qy} + A_{ix} M_{iy} - M_{ix} A_{qy} - M_{qx} A_{iy} + (A_{ix} A_{qy} - A_{qx} A_{iy}) = 0,$$

from which, by transvecting with $A^y_i A^x_h$ and using (3.23)(i), we find

(4 . 2 2)

$$C_{iqyx} A^y_i A^x_h - g_{kh} M_{qy} A^y_p + g_{qh} M_{iy} A^y_p - M_{iy} A^y_h g_{qp} + M_{qy} A^y_h g_{ip} + (g_{kh} g_{qp} - g_{qh} g_{ip}) = 0 .$$

Substituting (4.22) in (3.25) yields

(4.23) $$C_{iqyx} - g_{kh} M_{qy} A^y_p + g_{qh} M_{iy} A^y_p - M_{iy} A^y_h g_{qp} + M_{qy} A^y_h g_{ip} = 0 ,$$

which shows that the Weyl conformal curvature tensor of M^n vanishes. Thus we have

Theorem 4.3 : Let M^n, n≥4, be an totally real submanifold normal to the structure vector field $V^i$of a Sasakian manifold M^{2n+1} with vanishing contact Bochner curvature tensor. If the second fundamental tensors commute, then M^n is conformally flat.

REFERENCES

B.Y. Chen, *1996 . "Mean curvature and shape operator of isometric immersions in real space-forms"* Glasg. Math. J.38, 87-97.

Y.H.Kim, C.W.Lee and D.W.Yoon (2003) *"Shape operator of slant sub manifolds in Sasakian spaces,"* Bull. Korean Math. Soc.40, 63-76.

K. Matsumoto and I. Mihai, 1997. *"An obstruction to the existence of minimal totally real immersions in a locally conformal Kaehler space form"* Bull. Yamagata Univ. Natur. Sci. 14 (2), 83-87.

K.Yano and M.Kon, 1984, *"Structures on manifolds,"* World Scientific, .

A. Carriazo, Y.H. Kim and D.W. Yoon, 2004 *"Some inequalities on totally real Sub manifolds in locally conformal Kaehler space forms,"* J.Korean Math.Soc. 41(5), pp.795-808.

M.Kon, 1976 *" Totally real sub manifolds in Kaehler manifold,"* J.Diff. Geom.11(1976), 251-257.

B.Y.Chen & K.Ogiue, 1974 *"On totally real sub manifolds,"* Trans. Amer. Math. Soc. 193, 257-266.

C.S.Houh, 1973 *"Some totally teal minimal surface in l "* Proc. Amer. Math. Soc. 40, 240-244.

S.Bochner, 1949 *"Curvature and Betti numbers II,"* Ann. of Math. 50, 77-93.

D.E.Blair, 1975 *"On a geometric meaning of the Bochner tensor,"* Giometriae Dedicata, 4, 33-38.

K.Yano & S.Inshihara, 1974, *"Kaehlerian manifolds with constant scalar curvature whose Vochner curvature tensor vanishes."* Hokkaido Math. J. 3, 297-304.

R.S.Sinha and H.C.Lal, 1982, *"Totally submanifolds of a Kaehlerian manifold with Semi-symmetric metric F-Connections,"* Ind. Jour.pure appl. Math. 13(9):1018-1023.

K...Yano, 1976, *"Totally real submanifolds of a Kaehelrian manifold"*, J. Diff. Geom.,11, 351-359.

G.D. Ludden, M.Okumara & K. Yano, 1975, *"Totally real sub manifolds of a complex manifold."* Attl. Accad.Naz. Lincei Rend. Cl. Sci.,Fls. Math. Natur.