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HYPERNORMAL CURVES ON A HERMITIAN HYPERSURFACE

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ABSTRACT

Hypernormal curves on a Riemannian hypersurfaces and a Finsler subspace have been defined and studied by Singh (1968, 1972). Further, Behari and saxena (1956) have described the hyper - Darboux lines of a Kaehler manifold. Moreover, Weatherburn (1938) has explored some special features of the hypersurfaces, the hypersurface in Euclidean space of constant curvature and defined the theory on certain quadratic hypersurfaces in Riemannian space.

The present paper is devoted to the study of hypernormal curves on a Hermitian hypersurface and the equations representing a hypernormal curve of order 1 are obtained. Also, Some properties of this curve are investigated.

Keywords: Hermitian, Hypersurface, Hypernomal, Metric, Finsler subspace.

INTRODUCTION

Here, we shall firstly define the Hermitian space and give some fundamental, which are the pre-requisites to study such a space.

Let us assume that there is a self- conjugate positive definite metric:

$$ds^2 = g_{ij}\, dz\, dz^{j1} \quad \ldots\ldots \quad (1.1)$$

in the complex manifold C_n of dimension n.

If the fundamental metric tensor g_{ij} is hybrid, then we evoke such a metric a Hermitian metric' and the comples manifold equipped with this metric is called a Hermitian manifold (Yano and Bochner 1953), which will be denoted by H_n, and we shall always assume the adjointness of the indices.

1) We shall use the following range of the indices:

$i, j, k, \ldots\ldots = 1, 2, 3, \ldots\ldots, n+1$

$\bar{i}, \bar{j}, \bar{k}, \ldots\ldots = \bar{1}, \bar{2}, \bar{3}, \ldots\ldots, \overline{(n+1)}$.

$\alpha, \beta, \gamma, \ldots\ldots = 1, 2, 3, \ldots\ldots, n$.

$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \ldots\ldots = \bar{1}, \bar{2}, \bar{3}, \ldots\ldots, \bar{n}$.

Since the fundamental metric tensorg_{ij} is hybrid, therefore it's covariant components will satisfy the relation:

$$g^{ij} = \begin{pmatrix} 0 & g^{\lambda\bar{\mu}} \\ g^{\bar{\lambda}\mu} & 0 \end{pmatrix}, \text{.......................................}(1.2)$$

or

$$F_h^{\ i} F_k^{\ j} g^{kh} = g^{ij}, \text{.......................................}(1.3)$$

where $F_h^{\ i}$ is an almost complex structure.

Now, let us consider an analytic Hermitian hypersurface H_n of the embedding space H_{n+1}. If $(V^\alpha, V^{\bar{\alpha}}) \equiv (V^1, V^2, \ldots\ldots, V^n, V^{\bar{1}}, V^{\bar{2}}, \ldots\ldots, V^{\bar{n}})$ denote the co-ordinates of a point in H_n, then the equation of the Hermitian hypersurface (,i.e., H_n –hypersurface) may be written in the form:

$$Z^i = Z^i(V^\alpha);\ Z^{\bar{i}} = Z^{\bar{i}}(V^{\bar{\alpha}}) \text{.............................}(1.4)$$

Suppose that $g_{\alpha\bar{\beta}}$ is the fundamental metric tensor, then we have

$$g_{\alpha\bar{\beta}} = g_{i\bar{j}} B_\alpha^{\ i} B_{\bar{\beta}}^{\ \bar{j}}, \text{.............................}(1.5)$$

where

$$B_\alpha^{\ i} = \frac{\partial z^i}{\partial V^\alpha};\ B_{\bar{\beta}}^{\ \bar{j}} = \frac{\partial z^{\bar{j}}}{\partial V^{\bar{\beta}}}.$$

Let $(N^i, N^{\bar{i}})$ be the components of unit normal vector to the hypersurface, then

$$2 g_{i\bar{j}} N^i N^{\bar{j}} = 1 \text{.....................................}(1.6)$$

and

$$g_{i\bar{j}} N^i B_{\bar{\beta}}^{\ \bar{j}} = 0;\ g_{i\bar{j}} N^{\bar{j}} B_\alpha^{\ i} = 0 \text{.........................}(1.7)$$

These (n+1) equations determine the (n+1) components N^α of the unit normal $\bar{N}$.

If $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the components of the first curvature vectors with respect to H_{n+1} and H_n respectively, then we have from Behari and Saxena (1956):

$$q^i = B_\alpha^{\ i} p^\alpha + K_n N^i \text{...............................}(1.8)$$

and

$$q^{\bar{i}} = B_{\bar{\alpha}}^{\ \bar{i}} p^{\bar{\alpha}} + \bar{K}_n N^{\bar{i}}, \text{.....................}(1.9)$$

where the normal curvature $(K_n, \bar{K}_n)$ of the hypersurface is given by:

$$K_n = \Omega_{\alpha\beta} \left(\frac{dV^\alpha}{ds}\right)\left(\frac{dV^\beta}{ds}\right),$$

$$\bar{K}_n = \Omega_{\bar{\alpha}\bar{\beta}} \left(\frac{dV^{\bar{\alpha}}}{ds}\right)\left(\frac{dV^{\bar{\beta}}}{ds}\right).$$

and

Since λ_{**} is a unit vector field, we get

$$1 = \gamma^2 K_{(1)}{}^2 + K_{(1)}{}^2 K_{(2)}{}^2 \omega^2 \qquad (2.10)$$

in view of (2.1),(2.4) and (2.6c). The elimination of γ and ω from (2.6b),(2.8) and (2.10) will yield the equation for a hypernormal curve of order one.

From equation (2.8), we find

$$g_{\alpha\bar{\beta}}\, t^{\alpha}\, dv^{\bar{\beta}} \big/ ds = 0;\; g_{\bar{\alpha}\beta}\, t^{\bar{\alpha}}\, dv^{\beta} \big/ ds = 0,$$

which proves the following proposition:

A necessary condition that a curve be hypernormal (of order one) relative to congruence λ_{**} is that the tangential component to the hypersurface of the vector field λ_{**} is orthogonal to the curve.

After defining

$$\text{Cos}\alpha = \frac{g_{\alpha\bar{\beta}}\, t^{\alpha}\, \xi_{(1)}{}^{\bar{\beta}} + g_{\bar{\alpha}\beta}\, t^{\bar{\alpha}}\, \xi_{(1)}{}^{\beta}}{2\sqrt{\left(g_{\alpha\bar{\beta}}\, t^{\alpha}\, t^{\bar{\beta}}\right)}\sqrt{\left(g_{\alpha\bar{\beta}}\, \xi_{(1)}{}^{\alpha}\, \xi_{(1)}{}^{\bar{\beta}}\right)}}$$

and

$$\text{Cos}\phi = \frac{g_{\alpha\bar{\beta}}\, t^{\alpha}\, \xi_{(2)}{}^{\bar{\beta}} + g_{\bar{\alpha}\beta}\, t^{\bar{\alpha}}\, \xi_{(2)}{}^{\beta}}{2\sqrt{\left(g_{\alpha\bar{\beta}}\, t^{\alpha}\, t^{\bar{\beta}}\right)}\sqrt{\left(g_{\alpha\bar{\beta}}\, \xi_{(2)}{}^{\alpha}\, \xi_{(2)}{}^{\bar{\beta}}\right)}}$$

and multiplying (2.8) respectively by $\left(2\, g_{\alpha\bar{\beta}}\, \xi_{(1)}{}^{\bar{\beta}}\right)^{1/2}$ and $\left(2 g_{\alpha\bar{\beta}}\, \xi_{(2)}{}^{\bar{\beta}}\right)^{1/2}$, we deduce

$$\sin\theta \cos\alpha = \gamma K_{(1)} + w\,\{K_{(1)}\, d \log(K_{(1)}/K^{*}{}_{(1)})/ds - K_{(n)}\Omega_{\alpha\beta}\, \xi_{(1)}{}^{\beta}\, dv^{\alpha}/ds\} \qquad (2.11)$$

and

$$\text{Sin}\theta\, \text{Cos}\phi = w(K_{(1)} K_{(2)} - K_{(n)}\Omega_{\alpha\beta} dv^{\alpha}/ds) \qquad (2.12)$$

where we have used (2.3).

Defining v = $K_{(1)}/K_{(n)}$ and eliminating γ and w from (2.6b),(2.11) and (2.12), we get

$$(\text{Sin}\theta\, \text{Cos}\alpha - v\, \text{Cos}\theta)\,(vK_{(2)} - \Omega_{\alpha\beta}\, \xi_{(2)}{}^{\beta}\, dv^{\alpha}/ds) = \text{Sin}\theta\, \text{Cos}\phi\, \{dv^{\alpha}/ds - (1+v^2)\Omega_{\alpha\beta}\, \xi_{(1)}{}^{\beta}\, dv^{\alpha}/ds\}, \qquad (2.13)$$

where the relation

$K_{(1)}{}^{*2} K_{(n)}{}^2 = 1 + \frac{K_{(1)}{}^2}{K_{(n)}{}^2}$ has been used in the simplification.

3. PARTICULAR CASES

Now, we shall consider the solution of (2.13) in the following two particular cases:

Case (I): Let $\left(t^{\alpha}, t^{\bar{\alpha}}\right)$ be orthonormal to the first binoraml vector

$(\xi_{(2)}{}^{\alpha}, \bar{\xi}_{(2)}{}^{\bar{\alpha}})$. In particular, let $(t^{\alpha}, t^{\bar{\alpha}})$ be along the principal normal vector $(\xi_{(1)}{}^{\alpha}, \xi_{(1)}{}^{\bar{\alpha}})$. This implies $\mathrm{Cos}\phi=0$. Hence, we have either

$$K_{(1)}/K_{(n)} = \tan\theta . \mathrm{Cos}\alpha, \quad \text{.......(3.1)}$$

or

$$K_{(1)}.K_{(2)} = \Omega_{\alpha\beta}\Omega_{\gamma\varphi}(dv^{\alpha}/ds)(dv^{\beta}/ds)(dv^{\gamma}/ds)\xi_{(2)}{}^{\varphi} \quad \text{......(3.2)}$$

and their conjugates, where, we have used

$K_{(n)} = \Omega_{\alpha\beta}(dv^{\alpha}/ds)(dv^{\beta}/ds)$ in the later equation.

Case (II): Let the congruence λ_{**} be along the normal vector $\bar{N}$ of the hypersurface.

We have then $\mathrm{Cos}\theta=1$, $\mathrm{Sin}\theta=0$. Since the curve is non-geodesic, i.e., $v\neq 0$, equation (2.12) reduces to (3.2).

An especial feature of (3.2) is in fact the product of the curvatures (with respect to the hypersurface) of order one and two has been expressed in terms of the second fundamental tensor of the hypersurface.

Since, the curve is non-geodesic and non-asymptotic, we have the following proposition from (3.2):

A necessary and sufficient condition that the curvature of order two (with respect to the hypersurface) of a non-geodesic and non-asymptotic hypernormal curves of order one (with respect to normal congruence) to be zero is that the first binormal vector is conjugate with respect to its tangent vector.

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